LECTURE N° 5

delocalization of eigenvectors

$G = (V, E)$ finite $|V| = n$

v a unit eigenvector of A : $Av = \lambda v$ $\|v\|_2 = 1$

$$p \geq 2 \quad \|v\|_p = \Big(\sum_x |v(x)|^p\Big)^{1/p} \in \big[\underbrace{n^{1/p - 1/2}}_{\uparrow\, v = (\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}})}, 1\big] \quad \uparrow v = \delta_x$$

$$\|v\|_\infty = \sup_x |v(x)|$$

$$x \in V \quad |v(x)|^2 \leq \|P_{\ker(A-\lambda)} \delta_x\|_2^2 = \mu_G^{\delta_x}(\{\lambda\}).$$

One classical way to upper bound $\mu_G^{\delta_x}(\{\lambda\})$ is through the resolvent:

$$z \in \mathbb{C} \setminus \sigma(A) : \quad R(z) = (A - z)^{-1} \qquad R(z)_{xx} = \int \frac{d\mu_G^{\delta_x}(\lambda)}{\lambda - z} = \text{Cauchy-Stieltjes transform of } \mu_G^{\delta_x}$$

$$\lim_{\eta \to 0} \frac{1}{\pi} \operatorname{Im} R_{xx}(\lambda + i\eta) = \frac{d\mu_G^{\delta_x}}{d\,\mathrm{Lebesgue}}(\lambda) \quad \text{a.e.}$$

A weak result

Th (Chebyshev-Markov-Stieltjes) Let μ, ν be two probability measures on $[-b, b]$.

and assume that μ has a bounded density : $\left|\frac{d\mu}{d\,\mathrm{Lebesgue}}(\lambda)\right| \leq a \quad \forall \lambda \in \mathbb{R}$.

Assume $\int \lambda^k \, d\mu = \int \lambda^k \, d\nu$ for all $1 \leq k \leq n$

Then $$d_{KS}(\mu,\nu) \leq \frac{2\pi\, ab}{n-1}$$

where $d_{KS}(\mu,\nu) = \sup_{\lambda} |\mu(-\infty,\lambda) - \nu(-\infty,\lambda)|$

Idea of proof

$$(*) = \int 1_{(-\infty,x)}(\lambda)\, d\mu(\lambda) - \int 1_{(-\infty,x)}(\lambda)\, d\nu(\lambda)$$

approximate $1_{(-\infty,x)}$ by a polynomial expressed in terms of the orthogonal polynomials of μ. $\int P_k P_\ell \, d\mu = \delta_{k\ell}$ $\quad \deg(P_k) = k$

$$(*) \leq \frac{1}{\sum_{k=0}^{n} P_k(x)^2} \quad \cdots$$

Corollary

If $G=(V,E)$ is a d-regular finite graph with girth = length of the shortest cycle = ℓ then

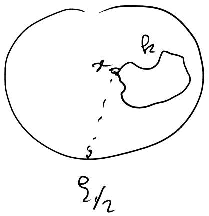
Brooks – Lindenstrauss

$\forall x \in V$ $$d_{KS}\left(\mu_G^{\delta_x}, \mu_{T_d}\right) \leq \frac{C\sqrt{d}}{\ell}$$ $C>0$ universal constant.

hence v unit eigenvector: $\|v\|_\infty \leq \sqrt{\frac{C\sqrt{d}}{\ell}}$.

the same statement holds (with a $\neq$ constant) if G is ℓ-tangle free.

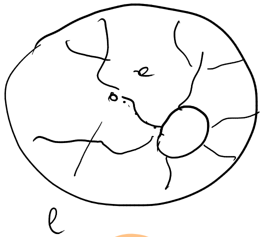
Proof: by assumption $\int \lambda^k \, d\mu_G^{\delta_x}$ = # closed walks from $x \to x$ of length k in G



if $k \leq 2\ell$ $\quad = \overline{}\ T_d$

$= \int \lambda^k \, d\mu_{T_d}$.

Def a graph $G=(V,E)$ is ℓ-tangle free if $\forall x \in V$: $G|_{B_G(x,\ell)}$ has at most <u>one</u> cycle



Lemma let $1 \le d \le n-1$ with nd even and $\mathbb{G}(n,d) = \{d\text{-regular graphs on vertex set } \{1,\dots,n\} = [n]\}$. $d \ge 3$

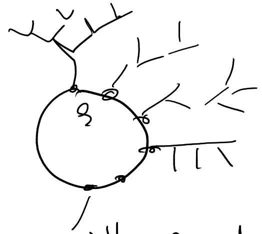
let $G \sim$ uniformly sampled on $\mathbb{G}(n,d)$

with probability tending to 1 as $n \to \infty$ G is ℓ-tangle free with

$$\ell = \left\lfloor \frac{1}{4} \frac{\log n}{\log (d-1)} \right\rfloor$$

<u>proof of Corollary for ℓ-tangle free graphs:</u>

H_{ℓ_2}: $d=3$ $\ell_2 \ge 3$



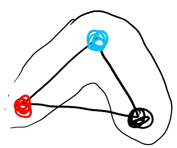
H_{ℓ_2} = infinite d-regular graph with one cycle of length ℓ_2.

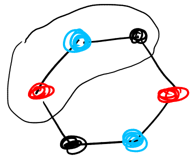
(**) $x \in V(H_{\ell_2})$ $\mu^{S_x}_{H_{\ell_2}}$ = is absolutely continuous with density bounded by $c\sqrt{d}$. ?

Def (covering of graphs) G_1, G_2 two <u>connected</u> graphs G_1 is a cover of G_2 if there exists a map $\varphi : V_1 \to V_2$ surjective such that

$\forall x \in V_1$ $\varphi|_{B_{G_1}(x,1)} : B_{G_1}(x,1) \to B_{G_2}(\varphi(x), 1)$ bijective.

φ: covering map

G_2 

 G_1

G_1 is a 2-cover of G_2

* if $|\varphi^{-1}(x)| = n$ is finite then $\forall y \in V_2$ $|\varphi^{-1}(y)| = n$

and G_2 is called a n-cover of G_1

The universal cover of G : T_G is a cover of G such that if H is a cover G then T_G is a cover of H.

T_G is a tree and is unique up to isomorphisms.

Rq : If G is a d-regular graph then T_G is the infinite d-regular tree.

Lemma If G_1 is a cover of G_2 and $\deg(x) \leq \Delta$ $\forall x \in G_1$ then

$R_i(z) = (A_i - z)^{-1}$. φ a cover s.t. $\varphi(u) = x$

$\forall y \in V_2$
$$\big(R_2(z)\big)_{xy} = \sum_{v:\, \varphi(v) = y} \big(R_1(z)\big)_{uv}$$

proof :
$$(A_2)^k_{xy} = \sum_{\substack{\text{walks in } G_2 \\ \gamma = (x_0, x_1, \dots, x_k) \\ x_0 = x}} (\mathbb{1}_{x_k = y}) = \sum_{\substack{\text{walks in } G_1 \\ \gamma = (u_0, \dots, u_k) \\ u_0 = u}} \mathbb{1}(\varphi(u_k) = y)$$
$$= \sum_{v:\, \varphi(v) = y} (A_1)^k_{uv}$$

Since $\|A_i\|_{2\to 2} \le \Delta$ if $|z| > \Delta$ then

$$R_i(z) = (A_i - z)^{-1} = -\sum_{q=0}^{\infty} \frac{A_i^{q}}{z^{q+1}} \Rightarrow (R_2(z))_{xy} = -\sum_{k=0}^{\infty} \frac{(A_2^{k})_{xy}}{z^{k+1}}$$

$$= \sum_{v\,:\,\varphi(v)=y} \frac{(A_1^{k})_{uv}}{z^{k+1}}$$

$$= \sum_{v\,:\,\varphi(v)=y} (R_1(z))_{uv}.$$

<u>Proof</u> of $(**)$: Enough to prove that $\left|\left(R_{H_k}(z)\right)_{xx}\right| \le C \quad \forall\, x \in V(H_k)$

i.e. $\lim_{\eta\to 0} \frac{1}{\pi} \operatorname{Im} R_{H_k}(\lambda + i\eta)_{xx}$ = density of $\nu_{H_k}^{\delta_x}$.

$$\left(R_{T_d}(z)\right)_{uv} = \left(R_{T_d}(z)\right)_{uu} \times \left(-\partial(z)\right)^{d_{T_d}(u,v)}$$

where $\partial(z) = \left(R_{T'_d}\right)_{oo}$ where T'_d is the infinite $(d-1)$-ary tree

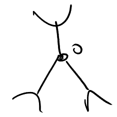
$d=3$

$$|R_{T_d}(z)| \le \frac{C}{\sqrt{d-1}} \qquad |\partial(z)| \le \frac{1}{\sqrt{d-1}}.$$

$$|R_{T_d}(z)_{uv}| \le C\left(\frac{1}{\sqrt{d-1}}\right)^{d_{T_d}(u,v)}$$

$$\left|\left(R_{H_k}(z)\right)_{xx}\right| \le \sum_{u:\,\phi(u)=o} \left|R_{\mathbb{T}_d}(z)\right|_{ou}$$

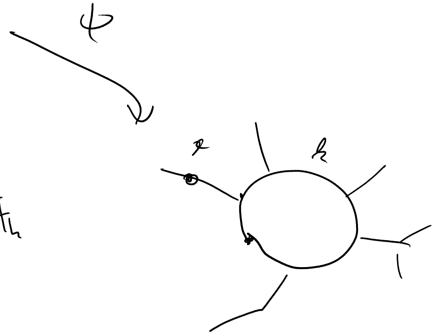
$\mathbb{T}_d$



$\phi(o) = x$

$$\le \sum_{u:\,\phi(u)=o} C\left(\frac{1}{\sqrt{d-1}}\right)^{d_{\mathbb{T}_d}(o,u)}$$

ϕ



H_k

$$\le 2\sum_{n=0}^{\infty} C\left(\frac{1}{\sqrt{d-1}}\right)^{a+kn}, \quad d\ge 3$$

$$\underline{\le C'}.$$

$d\geq 3$

* For uniform d-regular graphs on n vertices: it proves that if v a unit eigenvector then

$\|v\|_2 = 1$ $\qquad \|v\|_\infty \le \dfrac{C_d}{\sqrt{\log n}}$. This is a very weak

Much more is true Huang–Yau (21) with probability tending to 1: $\|v\|_\infty \le \dfrac{(\log n)^C}{\sqrt{n}} \lesssim \dfrac{1}{\sqrt{n}}$

The Gaussian eigenwave: universal BS limit of delocalized eigenvectors?

$G = (V,E)$ a graph with $|V| = \infty$ $\qquad \eta\left((A-\lambda)^2+\eta^2 1\right)^{-1} = \operatorname{Im} R(\lambda+i\eta)_{xy} = \Sigma_{xy}(\eta)$

fix $\lambda\in\mathbb{R}$

$R = (A-z)^{-1}$

$\operatorname{Im} R = \operatorname{Im}(z)\,|A-z|^{-2}$

Assume that $\boxed{\Sigma_{xy}(\eta) \xrightarrow[\eta\to 0]{} \Sigma_{xy}}$ $(*)$

Recall that Σ_{xx} = π. density of $\mu_G^{\delta_x}$ at λ.

Def if $\varphi \in \mathbb{C}^V$ we say that φ is a λ - eigenwave $\quad \underbrace{(A\varphi)(x)}_{= \sum_{y \sim x} \varphi(y)} = \lambda \varphi(x) \quad \forall x$

(beware φ is not assumed to be in $\ell^2(V)$)

$$\Sigma(\eta) \geq 0 \qquad \langle \varphi, \Sigma(\eta)\varphi \rangle \geq 0 \quad \forall \varphi \in \mathbb{C}^V.$$

by the Kolmogorov extension theorem, there exists a Gaussian centred random variable

$Y(\eta) = (Y(\eta)_x)_{x \in V} \in \mathbb{C}^V$ such that $\mathbb{E}\, Y(\eta) Y(\eta)^* = \Sigma(\eta)$.

Thanks to (*) $Y = \lim_{\eta \to 0} Y(\eta)$ is also well-defined $\quad \mathbb{E}\, Y Y^* = \Sigma$.

$$\mathbb{E}\left[\left| ((A-\lambda)Y(\eta))_x \right|^2 \right] = \langle \delta_x, (A-\lambda) \Sigma(\eta)(A-\lambda)\delta_x \rangle$$

$$= \eta \underbrace{\langle \delta_x, (A-\lambda)((A-\lambda)^2+\eta^2)^{-1}(A-\lambda)\delta_x \rangle}_{= T(\lambda)} \qquad \|T(\lambda)\|_{2\to 2} \leq 1$$

$$= \eta \underbrace{\langle \delta_x, (A-\lambda)^2 ((A-\lambda)^2+\eta^2)^{-1} \delta_x \rangle}$$

$$\leq \eta$$

Hence $\mathbb{E} \left| ((A-\lambda)Y)_x \right| = 0 \qquad \forall x \in V.$

Y is a.s. a λ-eigenwave.

If (G,o) is a unimodular random rooted graph with ρ,

If $\mathbb{E}(\Sigma_{oo}) = \mathbb{E}(Y_o^2) > 0$ $\left(\lim \mu_G^{\delta_o}(\lambda + i\eta) \xrightarrow[\eta \to 0]{} > 0 \right)$

then $\boxed{W_x = \frac{Y_x}{\sqrt{\mathbb{E}(\Sigma_{oo})}}}$, is the standard Gaussian on λ-eigenwave

$|V(G_n)| = n$

[It is conjectured that for nice graph sequences if $G_n \xrightarrow{BS} \rho$ and v_n is a unit eigenvector with eigenvalue $\lambda_n \to \lambda$ then $(\underbrace{G_n, v_n\sqrt{n}}, o) \xrightarrow{BS} \mathcal{L}aw(G, W, o)$]

Th Backhausz - Szegedy (2015) If $G_n \sim$ uniform d-regular on n vertices (nd even) v_n eigenvector $\|v_n\|_2 = 1$ and $\lambda_n \to \lambda$ then $(G_n, v_n\sqrt{n}, o) \xrightarrow{BS} \mathcal{L}aw(\mathbb{T}_d, \sigma W, o)$ for some $\sigma \in [0,1]$.

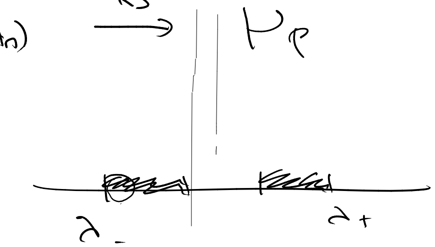
EXTREMAL EIGENVALUES of GRAPHS

① Benjamini - Schramm Convergence

$G_n = (V_n, E_n)$ is a sequence of graphs $|V_n| = n$

and $G_n \xrightarrow{BS} \rho = \mathcal{L}aw(G, o)$

We have seen that $\mu_{G_n} = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i(A_n)} \xrightarrow{KS} \mu_P$



$\forall \varepsilon > 0\ \exists \delta$

$\mu_P(\lambda_+ - \varepsilon, \lambda_+) > \delta$

$\Downarrow$

for all n large enough

$\mu_{G_n}(\lambda_+ - \varepsilon, +\infty) \geq \delta/2$

$\lambda_{\pm} = \sup_{\inf} \{ \lambda : \lambda \in supp(\mu_P) \}$.

$\lambda_1(A_n) \geq \lambda_2(A_n) \geq \dots \geq \lambda_n(A_n)$

$\forall k = o(n)$ $\liminf_{n\to\infty} \lambda_k(A_n) \geq \lambda_+$ and $\limsup_{n\to\infty} \lambda_{n+1-k}(A) \leq \lambda_-$

Do we have that $\lim_{n\to\infty} \lambda_k(A_n) \to \lambda_+$?

② Alon-Boppana lower bound

Th Alon-Boppana ('86) Let H be an infinite connected graph with $\max_{x\in H} \deg(x) \leq \Delta$ and H is quasi-transitive.

For any $\varepsilon > 0$ and $k \geq 1$ integer there exists $n_0 = n_0(\varepsilon, k, H)$ such that let $G = (V,E)$ be a finite graph covered by H. Then $\lambda_k(A_G) \geq \lambda_+ - \varepsilon$

$|V| = n$ $\lambda_1(A_G) \geq \dots \geq \lambda_n(A_G)$.

$\lambda_+ = \sup\{\lambda : \lambda \in \sigma(A_H)\}$

proof   Mohar (2010)

k fixed $B_G(S,k) = \{x \in V : d_G(x,S) \le k\}$

$S \subseteq V$ we have $|B_G(S,k)| \le |S| \cdot \Delta^k$.

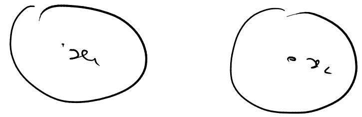
there exists $x_1, \dots, x_k \in V$ $B_G(x_i,k) \cap B_G(x_j,k) \neq 0 \quad \forall i \neq j$

$S = \bigcup_{i=1}^{k-1} B(x_i,k)$ $|S| \le k \cdot \Delta^k$ $B_G(S,k) \le k \cdot \Delta^{2k} = k \cdot n^{\frac{1}{4}} \le n \quad \varphi$

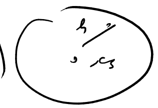
$$\boxed{k = \left\lfloor \frac{1}{4} \frac{\log n}{\log \Delta} \right\rfloor}$$

for n large enough

$$\tilde{A}_{xy} \begin{cases} = A_{xy} & \text{if } x, y \in S = \bigcup_{i=1}^{k} B(x_i,k) \\ = 0 & \text{otherwise} \end{cases}$$



$\tilde{A} = A_{|S}$ $\lambda_k(A) \ge \lambda_k(\tilde{A}) \ge \min_i \lambda_1(A_{|B(x_i,k)})$



Courant-Fisher

k even

$\|A\| = \|A^k\|^{\frac{1}{k}}$

$\|A\|_2 \ge \langle \varphi, A\varphi \rangle$

$\forall \varphi : \|\varphi\|_2 = 1$

$$\lambda_1(A_{|B(x_i,k)}) \ge \lambda_1\left((A_{|B(x,k)})^{k'}\right)^{\frac{1}{k'}}$$

$$\ge \langle \delta_x, A_{|B(x,k)}^{k'} \delta_x \rangle^{\frac{1}{k'}}$$

$$= \langle \delta_x, A^{k'} \delta_x \rangle^{\frac{1}{k'}}$$

if $k' \le 2k$

k' even

$\phi : V(H) \to V(G)$
$\varphi(u) = x$

$$= \Big(\text{no of closed walks of length } \ell_k' \; x \to x \text{ in } G\Big)^{1/\ell_k'}$$

$$\geq \Big(\text{———— in } H \quad \ell_k' \; u \to u\Big)^{1/\ell_k'}$$

$$= \langle \delta_u, A_H^{\ell_k'} \delta_u \rangle^{\frac{1}{\ell_k'}}$$

$\rho(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$ $\quad \geq \quad \rho(A_H) - \varepsilon$ for ℓ_k' long enough

$= \lim_{k\to\infty} \|A^{2k}\|^{\frac{1}{2k}}$

Gelfand' formula

[H is quasi-transitive $\Rightarrow$

$\langle \delta_u, A_H^{2n} \delta_u \rangle^{\frac{1}{2n}} \to \rho(A_H)$ uniformly in $u \in V(H)$]

If $\lambda_+ < \rho$ then apply the same argument to $(A + \Delta \cdot 1)$

$\lambda_+(A + \Delta 1) = \rho(A + \Delta 1)$

G d-regular $\quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ eigenvalues of $A = A|_G$

$|V| = n$

$\lambda_1 = d \qquad v_1 = \frac{1}{\sqrt{n}} \qquad Av_1 = d v_1$

$\lambda_2 \quad , v_2 \; ?$