# LECTURE N°4

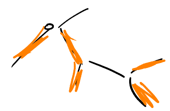
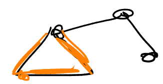
## Some positive answers

$(G,o)$ unimodular random rooted graph with law $\rho$: $\mu_\rho = \mathbb{E}[\mu_G^{\delta_o}]$

average quantum percolation

**Th** (B. Sen, Virág) $\mathrm{perc}(\mathbb{Z}^2, p)$ $p \leq p_c$ $\mu_{\mathrm{perc}(\mathbb{Z}^2,p)}$ is purely atomic

$p > p_c$ ——— has some non-trivial continuous part

**Def** a weighted graph $(G,\omega)$ $\omega : E \to \{0,1\}$  

is a line ensemble if $\forall x$ : $\sum_{y \sim x} \omega(x,y) \in \{0,2\}$

**Def** $(G,o)$ unimodular $L$ is an invariant line ensemble (ILE) if $(G,L,o)$ is unimodular

$$\forall f \in \mathcal{G}_{**}^{\to \mathbb{R}_+} \quad \mathbb{E}_\rho \sum_{v \in V} f(G,o,v) = \mathbb{E}_\rho \sum_{v \in V} f(G,v,o).$$

**Th** (B. Sen, Virág) $\forall \lambda \in \mathbb{R}$, $\mu_\rho(\{\lambda\}) \leq \mathbb{P}_\rho(o \notin L)\, \mu_{\rho'}(\{\lambda\})$

$\rho \sim$ law $(T,o)$ unimodular random rooted tree

let $L$ be an ILE.

$\rho'$ law of $(T,o)$ conditioned on $o \notin L$.

in particular: $(\mu_\rho)_{p.p}(\mathbb{R}) \leq \mathbb{P}_\rho(o \notin L)$.

$(\mu_\rho)_c(\mathbb{R}) \geq \mathbb{P}_\rho(o \in L)$

**Lemma** $\exists L \quad \mathbb{P}_\rho(o \in L) \geq \frac{1}{6} \frac{\left(\mathbb{E}\deg(o) - 2\right)^2}{\mathbb{E}\left[\deg(o)^2\right]}$.

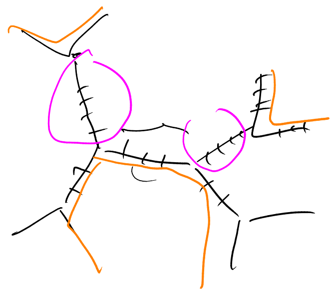
**Cor** $\rho \sim UGW(\pi) \quad \pi \neq \delta_2 \quad \mu_\rho$ contains a non-trivial continuous part if $\hat{\mu} > 1$

$\hat{\pi}(k) = \frac{(k+1)\pi(k+1)}{\sum \ell \pi(\ell)}$

$\hat{\mu} = \sum k \hat{\pi}(k)$

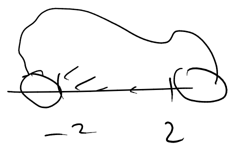
( recall that $\mathbb{P}_{UGW(\pi)}(|V(G)| = \infty) > 0 \iff \hat{\mu} > 1$ )
$\pi \neq \delta_2$.

open problem : $(\mu_\rho)_{p.p}(\mathbb{R}) = \inf_{L:\, ILE} \mathbb{P}_\rho(o \notin L)$ ?



$\mathrm{supp}((\mu_\rho)_{pp}) \subseteq [-2,2]$

$\mathrm{supp}(\mu_\rho) \supsetneq (-2,2)$



absolutely continuous part ?

**Th** (Coste - Salez) $\mu_\rho(\{0\}) = \max_{x \in [0,1]} M(x) = M(x^*)$ $\rho \sim UGW(\pi)$ (B, Lelarge, Salez)

$\frac{\mu_\rho((-\lambda,\lambda)) - \mu_\rho(\{0\})}{2\lambda} \longrightarrow \begin{cases} = 0 & \text{if } x_* \text{ is the unique max } M''(x_*) < 0 \\ > 0 & \text{if two maximizers of } M \end{cases}$

**Quantum percolation** : $\mu_G^{\delta_o}$ random probability measure

**Th** (Arras–B) $\quad P \sim GW(\pi) \quad d = \mathbb{E}[N]$ where $N \stackrel{d}{=} \pi$

+ Assume $\pi_0 d \leq C_0$.

(i) $\forall \varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that if

$$\mathbb{E}_\pi\left(\left(\frac{N}{d} + \frac{d}{N}\right) - 2\right)^{12} \leq C(\varepsilon)$$

then cond. on non-extinction $\mu_G^{\delta_o}$ has a.s. a non-trivial a.c. part and $\mathbb{E}\left(\mu_G^{\delta_o}\right)_{ac}(\mathbb{R}) \geq 1 - \varepsilon$.

+ If $\pi_0 = 0$, $\forall E \in (-2,2)$ there exists $C'(E) > 0$ such that

(ii) if

$$\mathbb{E}_\pi\left(\left(\frac{N}{d} + \frac{d}{N}\right) - 2\right)^{5} \leq C'(E)$$

then $\mu_G^{\delta_o}$ a.s. a.c on $(-E\sqrt{d}, E\sqrt{d})$ and a.e. positive density on $(-E\sqrt{d}, E\sqrt{d})$.

(i) applies $UGW(Poi(d))$ for $d \geq d(\varepsilon)$ $\quad N \sim Poi(d) \quad \frac{N}{d} \sim O\left(\frac{1}{\sqrt{d}}\right)$

**Ques** : threshold of quantum percolation smallest $d > 1$ such that $\mu^{\delta_o}_{UGW(Poi(d))}$ has a non trivial a.c part with proba $> 0$.

$d = e$ ? $\qquad d = 1{,}42\ldots$ ?

# Two tools for bounding eigenvalue multiplicity

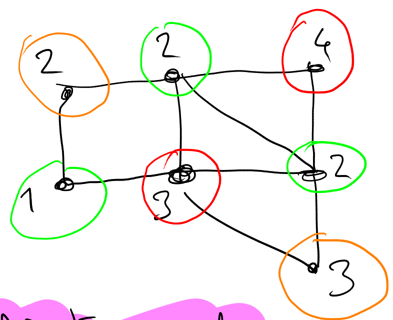
## ① Monotone labeling

**Def** Let $G=(V,E)$ be a graph and $\eta : V \to \mathbb{Z}$ labeling

(i) $v$ is a prodigy if $\exists w \sim v$, $w \neq v$, $\eta(w) < \eta(v)$ and $\forall u \sim w$, $u \neq v$, $\eta(u) < \eta(v)$

(ii) $v$ is level if $\forall w \sim v$ $\eta(w) \leq \eta(v)$ and not prodigy

(iii) $v$ is bad otherwise




**finite graphs**

**Th** $G$ be a finite graph. $\eta$ a labeling with $\ell$ level vertices and $b$ bad vertices. For any eigenvalue $\lambda$ with multiplicity $\dim(\ker(A-\lambda)) = m$ we have

$$m \leq b + \sum_j \ell_j$$

where $\ell_j = \dim(\ker(A|_{\text{level } j} - \lambda I))$

In part. $m_1 + \dots + m_k \le kb + \ell$

proof: $\ell_j(\lambda) = \dim(\ker(A|_{\text{level}_j} - \lambda I))$

$$m(\lambda) = \dim(\ker(A - \lambda I))$$

$$m(\lambda_1) + \dots + m(\lambda_k) \le kb + \sum_j \underbrace{\sum_{i=1}^{k} \ell_j(\lambda_i)}_{\le \text{ total number of level } j \text{ vertices.}}$$

$$\le kb + \ell$$

$|U| = n$

$E = \dim(\ker(A - \lambda I))$ $\quad \dim(E) = m$

$B$ = vectors vanishing on bad vertices $\quad \operatorname{codim}(B) = n - \dim(B)$

$$= \dim(B^{\perp}) = b.$$

$L_j$ = set of level $j$ vertices.

$E_j = \dim(\ker(A|_{L_j} - \lambda I))$ $\quad \dim(E_j) = \ell_j$

$E' = E \cap B \cap \bigcap_j E_j^{\perp}$ $\quad \dim(A \cap B) \ge \dim(A) - \operatorname{codim}(B)$

$$0 = \dim(E') \geq \dim(E) - \operatorname{codim}(B) - \sum_j \operatorname{codim}(G_j^{\perp})$$

$$= m - b - \sum_j \ell_j$$

We claim $E' = 0$. Indeed $f \in E'$

We prove that $f = 0$ by induction on $j \uparrow$. We assume that $f$ vanishes on vertices with label $i < j$

$f(x) = 0$ $x$ is bad.

$*$ $\varphi$ $v$ is a prodigy with label $j$ there exists $w \sim v$ with label $\ell(w) < j$ and $\forall u \sim w$ $u \neq v$, $\ell(u) < j$

in part: $f(w) = 0 = f(u)$

$$0 = \partial f(w) = \sum_{y \sim w} f(y) = f(v) + \sum_{\substack{u \sim w \\ u \neq v}} f(u) = f(v) + 0$$

$f(v) = 0$.

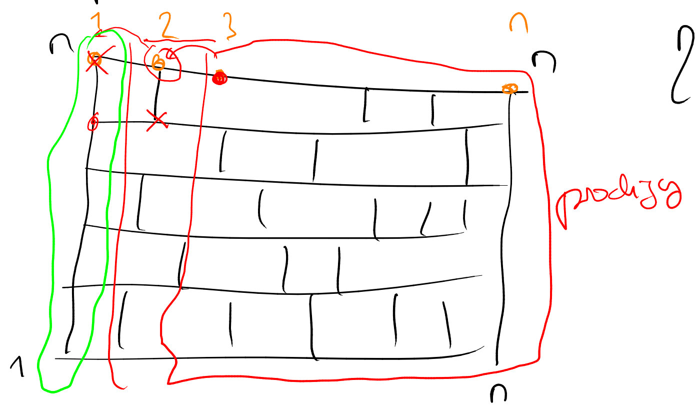
$*$ $L_j$ = level of vertices. $\partial L_j \subseteq B \cup \bigcup_{i<j} L_i \cup$ Prodigy with label $i \leq j$

$f|_{\partial L_j} = 0$ for recursion hypothesis

it implies $f|_{L_j}$ is an eigenvector $L_j$

but $f \in G_j^{\perp} \Rightarrow f|_{L_j} = 0$

example : vertical percolation




$$\eta^{(x,y)} = x$$

$$m(\lambda) \le n$$

$$\mu_G(\{\lambda\}) \le \frac{1}{n} \to 0 \qquad n \to \infty$$

bad

## Extension to unimodular graphs

$\rho \in \mathcal{P}_{uni}(\mathcal{G}_\bullet)$ $(\mathcal{M}, \tau)$ associated

von Neumann algebra with trace $\tau(T) = \mathbb{E}_\rho(\langle \delta_o, T \delta_o \rangle)$

$\mathcal{M} \ni P_S$ orthogonal projection onto a closed vector space $S$

$$\dim(S) := \tau(P_S) = \mathbb{E}_\rho(\| P_S \delta_o \|_2^2) .$$

**Th** let $(G,o)$ be a unimodular graph with law $\rho$ and $\eta : V \to \mathbb{Z}$ invariant coloring $(G,\eta,o)$ unimodular

$\ell = \mathbb{P}_\rho(o \text{ is level}) = \dim(\text{level vertices})$

$b = \mathbb{P}_\rho(o \text{ is bad}) = \dim(\text{bad vertices})$

$\ell_j = \dim(\ker(A_{|\text{level}_j} - \lambda))$

$\forall \lambda \quad \mu_\rho(\{\lambda\}) \le b + \sum_j \ell_j$

| and $\qquad \mu_p(\{\lambda_1\}) + \dots + \mu_p(\{\lambda_\ell\}) \le \frac{k}{2}b + e$.

proof see proof

**Th** (B, Sev, V.rég) $\quad \forall \lambda \in \mathbb{R}$, $\qquad \mu_p(\{\lambda\}) \le \mathbb{P}_p(o \notin L)\, \mu_{p'}(\{\lambda\})$

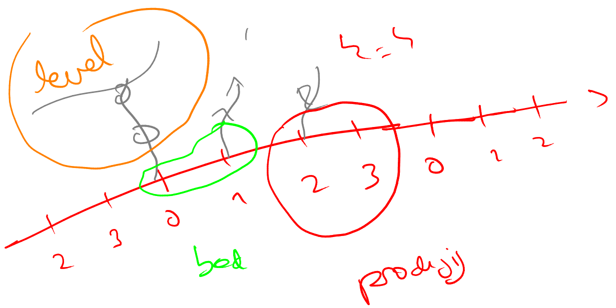
$\rho \sim$ law $(T, o)$ unimodular random rooted tree

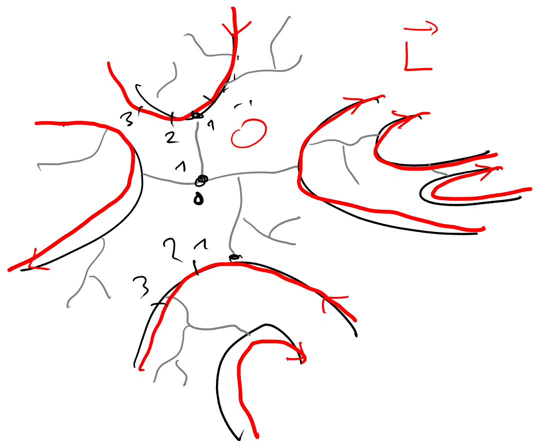
Let $L$ be an ILE. $\qquad$ $p'$ law of $(T,o)$ conditioned on $o \notin L$.

In particular: $\quad (\mu_p)_{p.p}(\mathbb{R}) \le \mathbb{P}_p(o \notin L)$.

$\qquad (\mu_p)_c(\mathbb{R}) \ge \mathbb{P}_p(o \in L)$

sketch of the proof :




$\eta : V \to \mathbb{Z}/k\mathbb{Z}$

$(\partial\eta)(u,v) = \eta(v) - \eta(u)$

$= \vec{L}(u,v)$

$\vec{L}(u,v) = \begin{cases} 1 & \text{if } (u,v) \in \vec{L} \\ -1 & \text{if } (v,u) \in \vec{L} \\ 0 & \text{otherwise} \end{cases}$

there are exactly $k$ such labeling

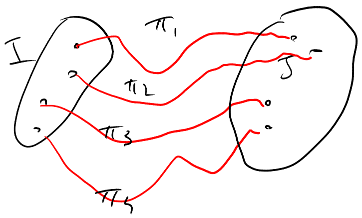
Let $\eta$ be a uniform $k$-labeling $\qquad \eta$ is an invariant labeling

## ② Minimal path matching (MPM)

**Def** $G = (V,E)$ be a finite graph $I = \{i_1, \dots, i_b\}$, $J = \{j_1, \dots, j_b\}$

$I \cap J = \emptyset$

a path matching $\pi = \{\pi_1, \dots, \pi_b\}$ from $I$ to $J$ $\quad \pi_i \cap \pi_j = \emptyset$ vertex disjoint




$$\pi_e = (u_{e,1}, \dots, u_{e,p_e})$$

$u_{e,1} = i_e$

$u_{e,p_e} = j_{\sigma(e)}$ where $\sigma \in S_b$ matching map

length of $\pi$: $|\pi| = \sum_{e=1}^{b} |\pi_e| = \sum_{e=1}^{b} p_e$

$\pi$ is a MPM from $I$ to $J$ if its length is minimal over path matchings from $I$ to $J$.

**Th** $G = (V,E)$ finite graph $|I| = |J| = b$ disjoint

Assume that MPM have the same matching map $\sigma$

If $|V| - \ell$ is the length of a MPM from $I$ to $J$ then

$$\sum_{i=1}^{r} (m_i - b)_+ \leq \ell$$

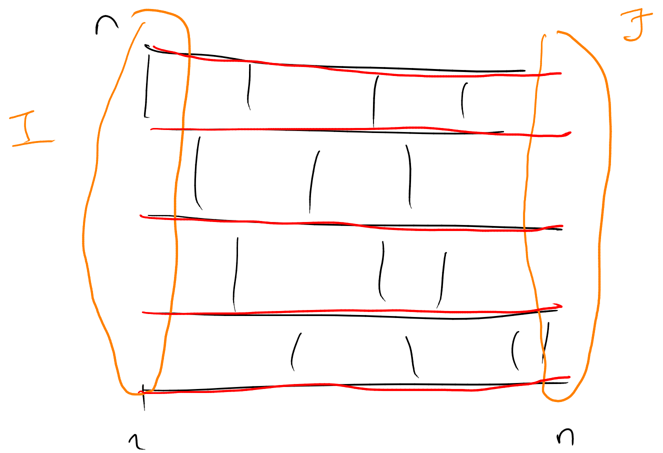
where $m_1, \dots, m_r$ are the eigenvalues multiplicities

$(x)_+ = \max(x, 0)$.

"$\deg(\Delta_b)$"

L In part: $m_1 + \cdots + m_k \leq \ell k b + \ell$

ex




$b = n$

$\ell = 0$

$m_i \leq n$ !

(Kim-Shader for trees — B, Son, Vries)

$|V| = n$

$I, J \subseteq V$  $(A - x)_{I,J}$  $A - x$ with rows $I$ and columns $J$ removed

$|I| = |J|$

$$P_{I,J}(A)(x) = \det (A - x)_{I,J} \in \mathbb{R}[x]$$

$$\Delta_b(A) = \mathrm{GCD}\left( P_{I,J}(A), \ |I| = |J| = b \right)$$

it is a monic polynomial  recall that $P \mid 0 \ \forall P \in \mathbb{R}[x]$

$$\deg\left(P_{I,I}(A)\right) = n - b \Rightarrow \deg(\Delta_b(A)) \leq n - b.$$

**Lemma** $A$ a symmetric matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ with multiplicities $m_1, \ldots, m_r$

1.

$$\Delta_b(A) = \prod_{i=1}^{r} (x - \lambda_i)^{(m_i - b)_+}$$

In part: $\deg(\Delta_b(A)) = \sum_{i=1}^{r} (m_i - b)_+$

proof $|V| = n$ $B(x) \in M_n(\mathbb{R}[x])$ $P_{I,J}(B) = \det(B_{I,J})$

$\Delta_b(B) = \mathrm{GCD}(P_{I,J}(B) : |I| = |J| = b)$

$B(x) = \left(B_1(x) \middle| B_2(x) \middle| \dots \middle| B_n(x)\right)$

$\Delta_b(A - x) = \Delta_b(A)$ with our previous notation.

$$\det\left(w_{11} B_1(x) + \dots + w_{n1} B_n(x), B_2(x), \dots, B_n(x)\right)_{1,J}$$

$1 \notin J$

$$= \sum_{j=1}^{n} w_{1j} \det(B_j, B_2, \dots, B_n)_{1,J}$$

is divisible by $\Delta_b(B)$

$1 \in J$

We deduce that $U, W \in M_n(\mathbb{R})$ $\Delta_b(B) \mid \Delta_b(UBW)$

Hence $U, W \in GL_n(\mathbb{R})$ $\Delta_b(B) = \Delta_b(UBW)$

If $A$ is symmetric $A = UDU^*$ $D$ diagonal $UU^* = 1$

$A - x = U(D - x)U^*$ $\Delta_b(A - x) = \Delta_b(D - x)$.

$$D = \begin{pmatrix} \lambda_1 \overbrace{\ddots}^{m_1} & & & 0 \\ & \lambda_1 & & \\ & & \lambda_1 \overbrace{\ddots}^{m_2} & \\ & & & \lambda_2 \\ & & & & \ddots \end{pmatrix}$$

$P_{I,J}(D - x) = 0$ if $I \neq J$

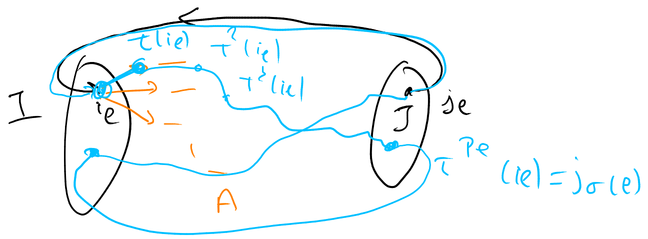
<u>proof of theorem</u>:

$\deg\left(\det(A - x)_{I,J}\right) \overset{?}{=} \ell$ assume matching map of NPN $\sigma = \mathrm{Id}$.

$B \in M_n(\mathbb{R})$

$1 \leq \ell \leq b$ $\quad B\,\delta_{j_\ell} = \delta_{i_\ell}$

$j \notin J \quad B\,\delta_j = \sum_{i \notin I} A_{ij}\,\delta_i$




$D =$ diagonal matrix with $D_{xx} = \begin{cases} 1 & \text{if } x \notin I \cup J \\ 0 & \text{otherwise} \end{cases}$

$$\det(B - xD) = \det\left((A - x)_{I,J}\right)$$

$$= \sum_{\tau \in S_n} \varepsilon(\tau) \prod_{v=1}^{n} \underbrace{(B - xD)_{v\tau(v)}}_{B_{v\tau(v)} \text{ if } \tau(v) \neq v,\ \in \{1,0\}} = \sum_{\tau \in S_n} \varepsilon(\tau)\, Q_\tau(x)$$

$$\overset{?}{=} \sum_{\pi \text{ matching paths from } I \text{ to } J} \underbrace{\varepsilon(\pi)}_{\in \{-1,1\}} \times \pm 1 \times \det\left((A - x)_{\pi,\pi}\right)$$

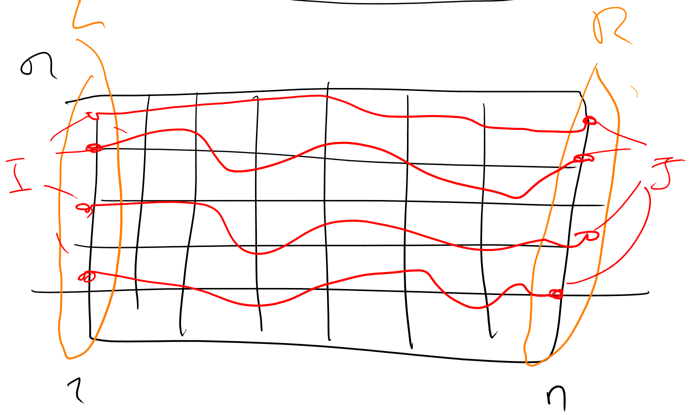
$$\deg\left(\det(A - e)_{\pi,\pi}\right) = n - |\pi|$$

$$= (-x)^\ell + \dots$$

if NPN is <u>unique</u>

$$\deg(\Delta_S(A)) \leq \ell .$$

## Application to percolation on $\mathbb{Z}^2$:




(Kesten 80)

$p > p_c$

unique MPM.

$p > p_c$

$|I| = |J| = b \geq \varepsilon n$

$|\Pi_*| \geq bn \geq \varepsilon n^2$

$\quad \shortparallel$

$n^2 - \ell \quad \Rightarrow \quad \ell \leq n^2(1-\varepsilon)$

$$m_1 + \dots + m_k \leq k \underbrace{\varepsilon n}_{b} + \underbrace{n^2(1-\varepsilon)}_{\ell} \leq kn + n^2(1-\varepsilon)$$

$$\mu_G(\lambda_1) + \dots + \mu_G(\lambda_k) \leq \frac{k}{n} + (1-\varepsilon)$$

$$\xrightarrow[n\to\infty]{} 1-\varepsilon < 1 .$$