LECTURE N°3

$G = (V, E)$ locally finite

$x \in V$ $\mu_G^{\delta_x}$ spectral measure at vector δ_x

$$\int \lambda^k \, d\mu_G^{\delta_x} = \langle \delta_x, A^k \delta_x \rangle$$

if degrees are uniformly bounded

(G, o) root o unimodular random rooted graph with law $\rho \in \mathcal{P}(\mathcal{G}_\bullet)$

$f : \tilde{\mathcal{G}}_{\bullet\bullet} \to \mathbb{R}_+$ $\boxed{\mathbb{E}_\rho \sum_{v \in V} f(G, o, v) = \mathbb{E}_\rho \sum_{v \in V} f(G, v, o)}$ mass transport principle

$\mu_\rho := \mathbb{E}_\rho \left[\mu_G^{\delta_o} \right]$ . . spectral measure

if G finite and $U(G) =$ law of $(G(o), o)$ then

$$\mu_{U(G)} = \mu_G = \frac{1}{|V|} \sum_i \delta_{\lambda_i}$$ empirical distribution of eigenvalues.

Lemma (everything shows at the root) Aldous-Lyons Let $f : \tilde{\mathcal{G}}_\bullet \to \mathbb{R}_+$ such that ρ-a.s. $f(G, o) = 0$ then ρ-a.s. $\forall v \in V$ $f(G, v) = 0$

ρ unimodular

$f(G, o) = \mathbb{1}((G, o) \in E)$

proof : $F : \tilde{\mathcal{G}}_{\bullet\bullet} \to \mathbb{R}_+$ $F(G, u, v) = f(G, u)$.

$$0 = \mathbb{E}_\rho \sum_v f(G, o) = \mathbb{E}_\rho \sum_v \underbrace{f(G, v)}_{\geq 0} \Rightarrow \rho\text{-a.s. } \forall v \in V \ f(G, v) = 0.$$

Continuity of spectral measure

th ρ unimodular $\rho = \mathcal{L}(G,o)$ $\rho \in \mathcal{S}_{uni}(\mathcal{G}_\bullet)$

(i) ρ-a.s. A = adjacency operator of G is essentially self-adjoint $\bar{A} = A^*$.

$\mu_\rho = \mathbb{E}_\rho[\mu_G^{\delta_o}]$ is well defined

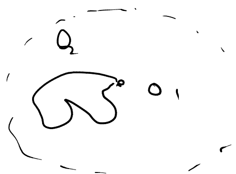
(ii) If $\rho_n \xrightarrow{w} \rho$ ρ_n unimodular then $\mu_{\rho_n} \xrightarrow{w} \mu_\rho$

Lemma The map $(G,o) \to \mu_G^{\delta_o}$ on $\{(G,o) : \forall v \ \deg_G(v) \leq \Delta\} = \tilde{\mathcal{G}}_\bullet(\Delta)$ is continuous for the loc topology

proof $\int \lambda^k d\mu_G^{\delta_o}(\lambda) =$ # closed walks from $o \to o$ of length k

In particular $(G_1,o_1)_h \simeq (G_2,o_2)_h$

then $\int \lambda^k d\mu_{G_1}^{\delta_{o_1}}(\lambda) = \int \lambda^k d\mu_{G_2}^{\delta_{o_2}}(\lambda)$ if $k \leq 2h$



$B(o_1, \frac{k}{2})$

$(G_n,o_n) \xrightarrow{loc} (G,o)$ there exists $h_n \to \infty$ such that

$(G_n,o_n)_{h_n} \simeq (G,o)_{h_n}$

In particular $\int \lambda^k \, d\mu_{G_n}^{\delta_{o_n}} \to \int \lambda^k \, d\mu_G^{\delta_o}$ $\forall k$.

since support $(\mu_{G_n}^{\delta_{o_n}}) \subseteq (-\Delta, \Delta)$ all these measures are characterized by their moments.

Proposition ρ unimodular there exists a probability measure μ_ρ on $\mathbb{R}$ such that $\forall (G_n)$ sequence of finite graphs such that $G_n \xrightarrow{BS} \rho$ (i.e. $U(G_n) \to \rho$) then $\mu_{G_n} \xrightarrow{w} \mu_\rho$

Lemma (Courant-Fischer min-max formula) If $A, B \in M_n(\mathbb{C})$ Hermitian

$$\mu_A = \frac{1}{n} \sum \delta_{\lambda_i(A)}$$

$$d_{KS}(\mu, \nu) = \sup_{t \in \mathbb{R}} \left| \mu(-\infty, t] - \nu(-\infty, t] \right|$$ Kolmogorov-Smirnov distance

$$d_{KS}(\mu_A, \mu_B) \leq \frac{\operatorname{rank}(A-B)}{n}$$

proof as an exercise... Courant-Fischer min-max formula:

$\lambda_1(A) \geq \dots \geq \lambda_n(A)$ $\qquad \lambda_k(A) = \sup_{\substack{H \subseteq \mathbb{C}^n \\ \dim(H) = k}} \min_{\substack{\varphi \in H \\ \|\varphi\|_2 = 1}} \langle \varphi, A\varphi \rangle$.

proof of the proposition: $|V(G_n)| = n$ $\qquad A_n = A(G_n)$

$$V_n(\Delta) = \{ v \in V_n : \underbrace{\forall u \in B_{G_n}(v,1)\ ,\ \deg(u) \le \Delta}_{\mathcal{E}_{(G_n, v)}} \} =$$

$$(A_n(\Delta))_{xy} = A_{xy} \cdot \mathbb{1}_{x,y \in V_n(\Delta)} .$$

$$\text{rank}(A_n - A_n(\Delta)) \le \text{Nb of non-zero rows of } A_n - A_n(\Delta)$$

$$\le n \, \mathbb{P}_{U(G_n)}\left(\mathcal{E}_{(G,o)}^c \right) = n \, \mathcal{E}_n(\Delta)$$

Since $U(G_n) \xrightarrow{w} \rho$ $\quad \mathcal{E}_n(\Delta) \to \mathcal{E}(\Delta) = \mathbb{P}_\rho\left(\mathcal{E}_{(G,o)}^c \right)$

$\rho = \text{Law}(G,o)$ locally finite $\quad \mathcal{E}(\Delta) \underset{\Delta \to \infty}{\longrightarrow} 0$

Hence $\forall \varepsilon > 0$ $\quad \mathcal{E}_n(\Delta) \le \varepsilon$ $\quad \forall$ for all n if Δ is well-chosen

m, n

$$d_{KS}(\underset{\mu_{G_m}}{\overset{\shortparallel}{\mu_{A_m}}}, \underset{\mu_{G_n}}{\overset{\shortparallel}{\mu_{A_n}}}) \le d_{KS}(\mu_{A_m(\Delta)}, \mu_{A_m}) + d_{KS}(\mu_{A_m(\Delta)}, \mu_{A_n(\Delta)})$$

$$+ d_{KS}(\mu_{A_n(\Delta)}, \mu_{A_n}).$$

$$\le 2\varepsilon + d_{KS}(\mu_{A_m(\Delta)}, \mu_{A_n(\Delta)})$$

$$\le 2\varepsilon + \varepsilon'$$

$U(G_n) \to \rho$ $\quad U(G_n(\Delta)) \to \rho(\Delta) = \text{Law}(G(\Delta), o)$ $\quad \overset{\mu_{A_n(\Delta)}}{\overset{\shortparallel}{\mu_{G_n(\Delta)}}} \to \mu_{\rho(\Delta)}$

$G_n(\Delta) = (V_n, E_n(\Delta))$ $\quad \underset{\{x,y\}}{\overset{\shortparallel}{e}} \in G_n \in G_n(\Delta)$ iff $x, y \in V_n(\Delta)$

by the continuity lemma

Thus $(\mu_{G_n})_n$ is a Cauchy sequence so they converge to $\mu_\rho \in \mathcal{P}(\mathbb{R})$.

To prove the theorem, we use the Von Neumann algebra associated to any unimodular probability measure.

Let $\mathcal{G}_\bullet$ be the set of labeled graphs with vertex set $V = \{0, 1, \dots\} = \mathbb{N}$ rooted at 0.

$T \in L^\infty(\mathcal{G}_\bullet, \mathcal{B}(\ell^2(V)), \rho)$ is a ρ a.s. random bounded operator on $\ell^2(V)$.

$\mathcal{M}$ is the subset of $L^\infty(\mathcal{G}_\bullet, \mathcal{B}(\ell^2(V)), \rho)$ which commute with the bijections on $V = \mathbb{N}$ $\quad T(G,o) = \lambda_\sigma^{-1} T(\sigma(G), o) \lambda_\sigma$ ρ-a.s.

$\sigma : V \to V$ bij. $\quad \lambda_\sigma(\delta_x) = \delta_{\sigma(x)}$

In part. $T(G,o) = T(G, \sigma(o))$ ρ a.s. invariant under re-rooting

$T \in \mathcal{M}$ $\quad \boxed{\tau(T) = \mathbb{E}_\rho \langle \delta_o, T \delta_o \rangle}$ faithful, normal, trace on $\mathcal{M}$.

$(\mathcal{M}, \tau)$

$\tau(TT^*) = 0$ iff $T = 0$

$$\tau(TT^*) = \mathbb{E}_\rho\left(\|T\delta_o\|_2^2\right) = \mathbb{E}_\rho\Big(\underbrace{\sum_v |\langle\delta_o, T\delta_v\rangle|^2}_{f(G,o)\,\geq\, 0}\Big)$$

If $\tau(TT^*) = 0$ then ρ-a.s. $f(G,o) = 0$ by the unimodularity lemma

ρ-a.s. $\forall v$ $f(G,v) = 0$ i.e. $\langle\delta_u, T\delta_v\rangle = 0$ $\forall u, v \in V$.

$$\tau(TS) = \mathbb{E}_\rho\left(\sum_v \langle\delta_o, T\delta_v\rangle\langle\delta_v, S\delta_o\rangle\right)$$

$$\overset{\text{unimodularity}}{=} \mathbb{E}_\rho\left(\sum_v \langle\delta_v, T\delta_o\rangle\langle\delta_o, S\delta_v\rangle\right)$$

$$= \tau(ST).$$

Continuity of atoms of the spectral measure

Theorem (Abért - Thom - Virág, Lück.) Let $\rho \in \mathcal{P}_{uni}(\mathcal{G}_\bullet)$. Let (G_n) be a sequence of finite graphs such that $G_n \xrightarrow{BS} \rho$.

Then $d_{KS}(\mu_{G_n}, \mu_\rho) \to 0$

(recall $d_{KS}(\mu,\nu) = \sup_{t\in\mathbb{R}} \left(|\mu(-\infty,t] - \nu(-\infty,t]| \right)$

$= \sup \left\{ \int f\,d\mu - \int f\,d\nu : \|f\|_{BV} \le 1 \right\}$

$\|f\|_{BV} = \sup_{t_1<t_2<\dots<t_n} \sum_{i=1}^{n} |f(t_i) - f(t_{i-1})|$

$\int |f'(t)|\,dt \quad f\in C^1.$

<u>Fact</u> $\mu_n \xrightarrow{KS} \mu \iff \begin{cases} \mu_n \xrightarrow{w} \mu. \\ \forall \lambda\in\mathbb{R} \quad \mu_n(\{\lambda\}) \to \mu(\{\lambda\}) \end{cases}$

<u>Corollary</u> If p is sofic $= \overline{\{ U(G) : G \text{ finite} \}}$ and λ s.t. $\mu_p(\{\lambda\}) > 0$
then λ is a <u>totally</u>-real algebraic integer : $\mathbb{A}$

i.e. $P(\lambda) = 0$ for some monic polynomial $\mathbb{Z}(x)$
with all roots real-valued

(or λ eigenvalue of a symmetric matrix
with integer coefficients)

<u>Conjecture</u> : $\forall\ p \in \mathrm{Sum}(\dot{S}_0)$ $\quad \mu_p(\{\lambda\}) > 0 \Rightarrow \lambda \in \mathbb{A}$

also open for groups

<u>Lemma</u> (Lück) let $\lambda\in\mathbb{R}$, $\Theta \in \mathbb{R}_+$ then there exists

a continuous function $\delta_{\geq 0} : \mathbb{R}_+ \to (0,1)$ with $\delta(0)=0$ such that
if G has degree bounded by Θ then $\forall \varepsilon>0$

$$\mu_G(\{\lambda\}) \leq \mu_G((\lambda-\varepsilon, \lambda+\varepsilon)) \leq \mu_G(\{\lambda\}) + \delta(\varepsilon).$$

(Also holds for $A \in M_n(\mathbb{Z})$ with $\|A\| \leq \Theta$).

<u>proof of thm</u> We restrict to bounded degree graphs: $\deg_{G_n}(x) \leq \Delta$ $\forall x$ and n.

It suffices to prove $\mu_{G_n}(\{\lambda\}) \to \mu_p(\{\lambda\})$ (because we already know that $\mu_{G_n} \xrightarrow{w} \mu_p$).

Portemonteau's lemma $\mu_n \xrightarrow{w} \mu \iff \forall\, O$ open $\underline{\lim}\, \mu_n(O) \geq \mu(O)$

$\iff \forall\, C$ closed $\overline{\lim}\, \mu_n(C) \leq \mu(C)$

We only have to prove $\underline{\lim}\, \mu_{G_n}(\{\lambda\}) \geq \mu_p(\{\lambda\})$

we have $\underline{\lim}\, \mu_{G_n}((\lambda-\varepsilon, \lambda+\varepsilon)) \geq \mu_p((\lambda-\varepsilon, \lambda+\varepsilon)) \geq \mu_p(\{\lambda\})$

but: $\mu_{G_n}(\lambda-\varepsilon, \lambda+\varepsilon) \leq \mu_{G_n}(\{\lambda\}) + \delta(\varepsilon)$

hence $\underline{\lim}\, \mu_{G_n}(\{\lambda\}) \geq \mu_p(\{\lambda\}) - \delta(\varepsilon)$. $\forall \varepsilon>0$

⊕

Proof of lemma: $\mu_G((\lambda-\varepsilon, \lambda+\varepsilon)) \le \mu_G(\{\lambda\}) + \delta(\varepsilon)$.

For $\lambda = 0$. $|V(G)| = n$

$I = (-\varepsilon, \varepsilon) \setminus \{0\}$ $\mu_G(I) \le \delta(\varepsilon)$?

let $N(I) = n\,\mu_G(I) = n^\circ$ of eigenvalues in I

We have: $\boxed{\prod_{\lambda_i \ne 0} \lambda_i \in \mathbb{Z} \setminus \{0\}.}$

$P(x) = \det(A - x)$

$$\Theta^n \cdot \varepsilon^{N(I)} \ge \Theta^{n - N(I)} \cdot \varepsilon^{N(I)} \geq \prod_{\lambda_i \ne 0} |\lambda_i| \ge 1$$

$$= \prod_{\lambda_i \in I} |\lambda_i| \cdot \prod_{\substack{\lambda_i \notin I \\ \lambda_i \ne 0}} |\lambda_i|$$

$$n \ln \Theta + n\,\mu_G(I) \ln(\varepsilon) \ge 0$$

$$\mu_G(I) \le \frac{\ln \Theta}{\ln(1/\varepsilon)} = \delta(\varepsilon)$$

Atoms and eigenvectors

ρ = unimodular measure on $\mathcal{G}_0$

$$\mu_\rho(\{\lambda\}) = \mathbb{E}_\rho\, \mu_G^{\delta_o}(\{\lambda\})$$

$$= \mathbb{E}_\rho \left[\| P_{E_\lambda} \delta_o \|_2^2 \right]$$

where E_λ = is vector space of λ eigenvectors.

$$= \tau\left(P_{E_\lambda} \right)$$

$(\mathcal{U}, \tau)$ associated UN algebra

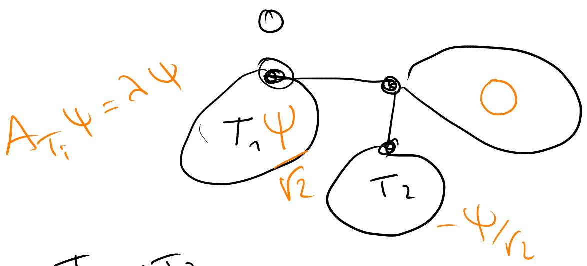
$$= \dim(E_\lambda)$$

L^2 Betti-numbers

$\ell^2(V) \ni \psi$ is an eigenvector $A\psi = \lambda\psi$ then

$\|\psi\|_2 = 1$

$$\mu_G^{\delta_o}(\{\lambda\}) \geq \langle \psi, \delta_o \rangle^2$$



$T_1 \simeq T_2$

Th (Bors – Salez) $\mathbb{A}$ = { totally-real algebraic integers }

$= \{ \lambda \in \sigma(A) : A = A_T$, T finite tree $\}$

Cor If $\rho \sim UGW(\pi)$ and $\mathrm{supp}(\pi) \supseteq \{1,2,\dots\}$

then $\mathrm{supp}\left((\mu_\rho)_{p.p} \right) = \mathbb{A}$

$$\hat\pi(k) = \frac{(k+1)\,\pi(k+1)}{\sum \ell\,\pi(\ell)}$$

$\mathrm{supp}(\hat\pi) = \mathbb{N}$

T_1

Explicit computations on atoms:

Th (Bordenave, Lelarge, Salez) $\quad \rho \sim UGW(\pi) \qquad \varphi(x) = \sum x^k \pi(k)$

$$\mu_\rho(\{0\}) = \max_{x \in [0,1]} M(x)$$

$$M(x) = \varphi'(1)\, x \bar{x} + \varphi(1-x) + \varphi(1-\bar{x}) - 1 \quad ; \quad \bar{x} = \frac{\varphi'(1-x)}{\varphi'(1)}$$

It gives some example and that $\mu_\rho(0) > 0$ but $\pi(0) = \pi(1) = \dots = \pi(k) = 0$.

Th (Salez) $\quad i(G) = \inf \left\{ \frac{|\partial S|}{|S|} : S \subseteq V(G),\ 0 < |S| < \infty \right\}$

If (T,o) is unimodular random tree with law ρ and p.s. $i(T) \geq i$, $\deg(o) \leq \Delta$

then $\mu_\rho(\{\lambda\}) \leq \frac{1}{\tau(\lambda)} \left(1 - \frac{\tau(\lambda)\, i}{\Delta} \right)_+$

$$\tau(\lambda) = \inf_{T \text{ finite}} \{ |V(T)| : \lambda \in \sigma(A_T) \} \quad \text{tree complexity}$$

Consequently: $\Lambda_{pp}(p) = \mathrm{supp}((\nu_p)_{p.p}) \subseteq \{ \lambda \in A : \tau(\lambda) \leq \frac{\Delta}{i} \}$

Quantum percolation

Anderson (56) de Gennes – Lafore – Millot (58).

(CP) Classical percolation: $p = \mathrm{Law}(G, o)$ unimodular random rooted graph $= (\mathrm{perc}(\mathbb{Z}^d, p)(o), o)$

$\mathbb{P}_p(|N(G)| = \infty) > 0.$

$\mu = \mu_{pp} + \mu_c = \mu_{pp} + \mu_{sc} + \mu_{ac}$

(AQP) average quantum percolation: $\mu_p = \mathbb{E}_p[\mu_G^{\delta_o}]$ has a non-trivial continuous part

(QP) quantum percolation: $\mathbb{P}_p((\mu_G^{\delta_o})_c(\mathbb{R}) > 0) > 0$

$$QP \Rightarrow AQP \Rightarrow CP$$

Fujii

If $V(G) < \infty$ a.s. then $\mu_G^{\delta_o}$ is purely atomic

Some positive results

$\mathrm{perc}(\mathbb{Z}^d, p)$ there exists $0 < p_c(d) < 1$ such that if $p < p_c$ a.s. no ∞ connected component if $p > p_c$ a.s. $\exists!$ ∞ connected component.

Th (B. Sen, Virág) $d=2$ If $p \leq p_c$ then μ_p is purely atomic

$p > p_c$ then μ_p has a non trivial cont. part

$d \geq 3$ completely open, for existence of an a.c. part also part