LECTURE N° 2

SUMMARY

$G=(V,E)$ locally finite : $\deg(x)<+\infty$, $\forall x\in V$

$C_c(V)\subseteq \ell^2(V)$ compactly supported vectors of $\ell^2(V)$

$$\boxed{(A\varphi)(x) = \sum_{e=\{x,y\}} \varphi(y)}, \quad x\in V.$$

A essentially self-adjoint thus we can define:

$\varphi\in C_c(V)$ $\quad \mu_G^{\varphi}(E) = \langle \varphi, 1\!\!1_E(A)\varphi\rangle$ $\quad E$ Borel in $\mathbb{R}$.

If A is bounded operator $\quad \int \lambda^k \, d\mu_G^{\delta_x}(\lambda) = \langle \delta_x, A^k \delta_x\rangle$

$=$ nb of closed walks in G of length k starting at x.

If G is finite $\quad \mu_G^{\varphi} = \sum_i \delta_{\lambda_i} \langle \varphi, u_i\rangle^2$ where λ_i are the eigenvalues and (u_i) O.N basis of eigenvectors

$$\boxed{\mu_G = \frac{1}{n}\sum_i \delta_{\lambda_i} = \frac{1}{|V|}\sum_{x\in V} \mu_G^{\delta_x}}$$

(average spectral measure)

$G=\mathrm{Cay}(\Gamma,S)$ $\quad G$ is transitive $\quad \mu_G^{\delta_x}$ does not depend on x

$o\in$ it $\quad \boxed{\mu_G := \mu_G^{\delta_o}}$ $\quad$ Plancherel measure

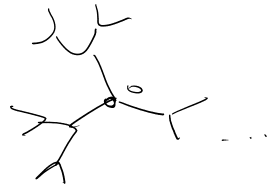
Examples of spectral measures of Cayley graphs

$\mathbb{Z}$  $\mathrm{Cay}(\mathbb{Z}, + -)$ bi-infinite path

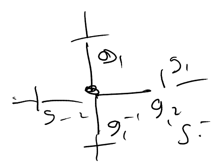
$$d\mu_{\mathbb{Z}}(\lambda) = \frac{1}{\pi\sqrt{4-\lambda^2}} 1_{|\lambda|\le 2}\, d\lambda .$$ arcsine law

Euclidean lattice $\mu_{\mathbb{Z}^d} = \underbrace{\mu_{\mathbb{Z}} * \dots * \mu_{\mathbb{Z}}}_{d}$ usual convolution

$\mathbb{T}_d$ = infinite d-regular tree



$d=2q$ Cayley graph of $\mathbb{F}_q$ with its natural set of generators.



$\mathbb{F}_2 \Rightarrow \mathbb{T}_4$. | $g_1, g_2, g_1^{-1}, g_2^{-1}$

[$\mathbb{T}_d$ = Cayley graph of $\mathbb{Z}/2\mathbb{Z} * \dots * \mathbb{Z}/2\mathbb{Z}$ with its natural free generators.

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$d\mu_{\mathbb{T}_d} = \frac{d}{2\pi} \frac{\sqrt{4(d-1) - x^2}}{d^2 - x^2} 1_{|x| \le 2\sqrt{d-1}}\, dx$$

Kesten 59

Kesten–McKay formula.

$\pi = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$ $= \pi \boxplus \dots \boxplus \pi$

$= \mu_{\mathbb{Z}/2\mathbb{Z}}$

Rq Note $\mu_{\mathbb{Z}} = \pi \boxplus \pi$ arcsine law.

Lamplighter group

Grigorchuk – Żuk (95') Lehner – Neuhauser – Woess.

give examples of Plancherel measure μ_G which is purely atomic

+ $\mu_G(\{0\})$ any number in $(0,1)$

disprove a conjecture of Atiyah Austin – Lehner/Wagner

Γ = finitely generated group "walk space"

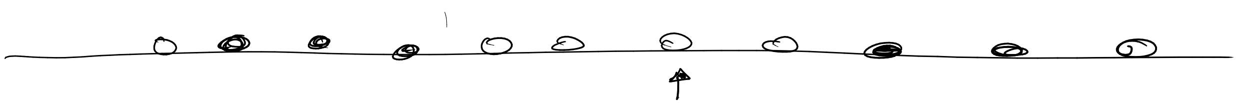
$L = (\mathbb{Z}/_{n\mathbb{Z}}, +)$ "lamp space"

$\Lambda = L \wr \Gamma$ lamplighter group. on the set $L^{\Gamma} \times \Gamma$

$(\eta, x) \in \Lambda$ $\eta \in L^{\Gamma}$ configuration $\eta : \Gamma \to \Lambda$

$x \in \Gamma$ position of the lamplighter

$\Gamma = \mathbb{Z}$ $n=2$



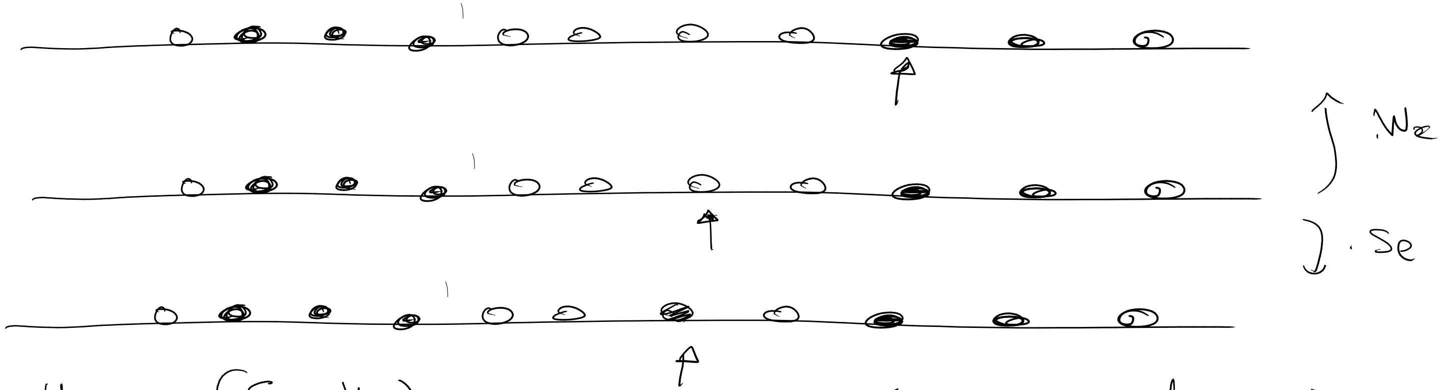
e = unit of Γ

$(\eta, x) \cdot (\eta', x') = (\eta + \theta_x \eta', x \cdot x')$

$\varepsilon = (\underline{0}, e)$ unit of Λ $(\theta_x \eta)(y) = \eta(x^{-1}y)$ $\theta_x \eta$ shifts the configuration η by x

$W_x = (\underline{0}, x)$ $x \in \Gamma$ $S_e = (\ell \delta_e, e)$ where $\delta_o(y) = \mathbb{1}(y = o)$.

$(\eta, x) W_y = (\eta, xy)$ $(\eta, x) \cdot S_e = (\eta + \ell \delta_x, x)$



Exercise that (S_e, W_x), $e \in L$ and $x \in D$ generate Λ

D = generating set of Γ finite symmetric

$SW = \{ S_e \cdot W_x, \ e \in L, \ x \in D \}$ is also a generating set of Λ

$WS = \{ W_x S_e, \quad \text{''} \quad \}$

"$\mu_{WS} = \mu_{Cay(\Lambda, WS)}$"

$SWS = \{ S_e W_x S_{e'}, \ e, e' \in L, \ x \in D \}$

$G = Cay(T, D)$

Theorem (LNW) $p = 1/n$ $\quad perc'(G,p)$ site percolation on T with parameter p

$$\mu_{SW}(\cdot/n) = \mu_{WS}(\cdot/n) = \mu_{SWS}(\cdot/n^2) = \mathbb{E}\left[\mu^{\delta_e}_{perc'(G,p)}\right].$$

$\mu(\cdot/t)$ is the push-forward of μ by the map $x \to x/t$

proof $\quad \nu = \mu_{WS}(\cdot/n) \qquad \mu = \mathbb{E}\left[\mu^{\delta_o}_{perc'(G,p)}\right].$

it is enough to prove that μ and ν have the same moments.

W_k = set of closed walks of length k from e to e in Cay $(\Gamma, D) = G$

= set of $\gamma = (\gamma_0, \dots, \gamma_k) \in \Gamma^{k+1}$ $\{\gamma_t, \gamma_{t+1}\} \in E(G)$. $t = 0, \dots, k-1$

the range of γ is the set $V(\gamma) = \{\gamma_t : t = 0, \dots k\}$

$$\int \lambda^k d\mu^{\delta_e}_{perc'(G,p)}(\lambda) = \sum_{\gamma \in W_k} \prod_{t=0}^{k} 1\!\!1(\gamma_t \text{ is open}).$$

$$\mathbb{E} \int \lambda^k d\mu^{\delta_e}_{perc'(G,p)} = \int \lambda^k d\mu(\lambda) = \sum_{\gamma \in W_k} p^{|V(\gamma)|} = \sum_{\gamma \in W_k} n^{-|V(\gamma)|}.$$

let $d = |D|$ A adjacency operator of $\mathrm{Cay}(\Lambda, WS)$ $P = \frac{A}{dn}$ is the corresponding transition kernel of the SRW on $\mathrm{Cay}(\Lambda, WS)$.

$\nu = \mu_{WS}(\cdot/n)$ $$\int \lambda^k d\nu = d^k \, \mathbb{P}_\varepsilon(S_k = \varepsilon)$$ $S_0 = \varepsilon, S_1, \dots, S_k$ SRW

$S_t = (\eta_t, \gamma_t)$ η_t = configuration of lights

γ_t = position of the lamplighter.

$S_t = X_1 \dots X_t = (\eta_t, \gamma_t)$

(X_t) iid $X_t = W_{x_t} \cdot S_{e_t}$ with x_t uniform on D $\perp$ e_t uniform on L

$\gamma = (\gamma_0, \dots \gamma_t)$ is a SRW on G independent of S_{e_t}'s.

$$\eta_t(y) = \begin{cases} \eta_{t-1}(y) & \text{if } y \neq \gamma_t \\ \eta_{t-1}(y) + e_t & \text{if } y = \gamma_t. \end{cases}$$

It follows $S_\ell = \varepsilon$ iff $\gamma_\ell = e$ and

$\forall x \in V(\gamma)$ at the last passage time of x, t_x we have

$$\eta_{t_x - 1}(x) + \ell_{t_x} = 0.$$

This happens with proba $1/n = p$ independently of the rest

$$\mathbb{P}_\varepsilon (S_\ell = \varepsilon) = \frac{1}{d^\ell} \sum_{\gamma \in W_\ell} p^{|V(\gamma)|}$$

Hence

$$= \frac{1}{d^\ell} \int \lambda^\ell \, d\mu(\lambda)$$

Corollary Grigorchuk Żuk $\Gamma = \mathbb{Z}$ and $n = 2$ $L = \mathbb{Z}/2$ $p = 1/2$

$$\mu_{WS}(1/2) = \sum_\ell p^\ell (1-p) \, \mu_{L_\ell}$$

L_ℓ

it is purely atomic !

$$\mu_{L_\ell} = \frac{1}{n} \sum_{\ell=1}^{n} \delta_{\cos\left(\frac{2\pi k}{n}\right)}$$

$G = \mathbb{T}_d$ and $n = 2$ $\mu_{WS}(3/4)$ has an explicit form..

quasi-transitive graphs colored Cayley graphs $G = \mathrm{Cay}(\Gamma, S)$

$a_g \in M_r(\mathbb{R})$

$$A_G = \sum_{g \in S} \lambda(g) \qquad A = \sum_{g \in S} a_g \otimes \lambda(g) \in B(\mathbb{C}^r \otimes \ell^2(V)).$$

is the adjacency of a quasi-transitive graph on V.

$V = \Gamma \times \{1, \dots, r\}$.

Spectral measure of unimodular graphs

Benjamini – Schramm topology :

Def (G,o) rooted graph : G : connected graph, $o \in V(G)$ root

$t \in \mathbb{N}$ $(G,o)_t$ rooted graph spanned by vertices at distance $\leq t$ from o.

(G_1,o_1) and (G_2,o_2) are isomorphic if $\phi : V_1 \to V_2$ bijection
$\phi(G_1) = G_2$ $\phi(o_1) = o_2$.

$\tilde{\mathcal{G}}_\bullet$ = set of eq. classes of locally finite rooted graphs.
unlabeled locally __________ .

V countable $\mathcal{G}_\bullet(V,o)$ = set of locally finite rooted graphs with vertex set V and root o.

Rq There is a canonical way to label the vertices of an unlabeled graph

If g, g' are in $\tilde{\mathcal{G}}_\bullet$ $\quad d_{loc}(g,\tilde{g}) = \sum_{t=0}^{\infty} 2^{-t} \mathbb{1}(g_t = g'_t)$

(G,o) $\qquad \mathcal{G}_\bullet(V,o)$

Lemma $(\tilde{\mathcal{G}}_\bullet, d_{loc})$ separable complete metric space
$(\mathcal{G}_\bullet(V,o), d_{loc})$ __________ .

$\mathcal{P}(\tilde{\mathcal{G}}_\bullet)$ proba measures on $\tilde{\mathcal{G}}_\bullet$ equipped with the weak topology

$\mathcal{P}(\mathcal{G}_\bullet(V,o))$

$P, P_n \in \mathcal{P}(\tilde{\mathcal{G}}_\bullet)$ $\quad P_n \xrightarrow{w} P$ if and only if $\forall t \geq 0$, $g \in \tilde{\mathcal{G}}_\bullet$

$P_n(g_t) \longrightarrow P(g_t)$.

$$P_n(g_t) = \mathbb{E}_P\left[\mathbb{1}((G,o)_t = g_t)\right].$$

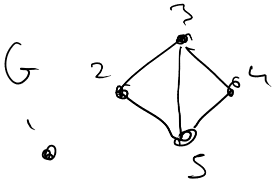
For a finite graph $G = (V,E)$ $\quad v \in V \quad G(v)$ G restricted to the connected component of v

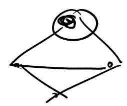
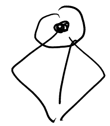
$g(v) =$ eq. class of $(G(v), v)$

$\in \tilde{\mathcal{G}}_\bullet$.

$$U(G) = \text{Law of } g(o) \text{ with } o \text{ uniform on } V$$

$$= \frac{1}{|V|}\sum_{v \in V} \delta_{g(v)} \in \mathcal{P}(\tilde{\mathcal{G}}_\bullet).$$

G 

$g(1) =$  $\quad g(2) = g(4) =$  $\quad g(3) = g(5) =$ 

Def a sequence of finite graphs (G_n) converges in the Benjamini-Schramm sense to $\rho \in \mathcal{P}(\tilde{\mathcal{G}}_\bullet)$ if $U(G_n) \xrightarrow{w} \rho$.

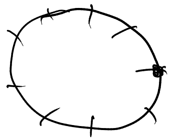
In other words $G_n \xrightarrow{BS} \rho$ is for any $t \geq 0$ and $g \in \tilde{S}_\bullet$

$$\frac{1}{|V(G_n)|} \sum_{v \in V(G_n)} \mathbb{1}\left((G_n(v),v)_t \simeq g_t \right) \longrightarrow \mathbb{P}_g\left(((G,o)_t = g_t \right).$$

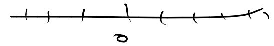
By extension if $\rho \in \mathcal{P}(S_\bullet(V))$ we say that $G_n \xrightarrow{BS} \rho$ if G_n converges to $\tilde{\rho}$ where $\tilde{\rho} \in \mathcal{P}(\tilde{S}_\bullet)$ ...

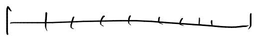
Examples

G_n



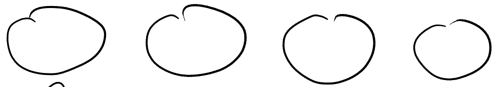
$$G_n \xrightarrow[n\to\infty]{BS} \delta_{(\mathbb{Z},0)}$$





L_n

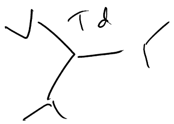
$$L_n \xrightarrow{BS} \delta_{(\mathbb{Z},0)}$$



G_n $\quad C'_n$ $\quad C''_n$ $\quad \dots$

$$\xrightarrow[n\to]{} \delta_{(\mathbb{Z},0)}$$

+ G_n d-regular graph on n vertices and $\gamma_n \to \infty$ girth = length of the shortest cycle



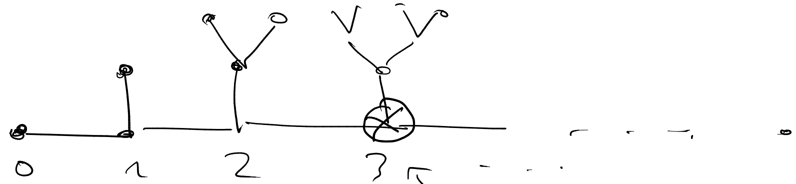
$$(G_n(x),x)_t \simeq (T_d, o)_t \quad \text{is} \quad t \leq \gamma_n.$$

$$U(G_n) \longrightarrow \delta_{(T_d,o)}$$

+ G_n

$G_n = (T_d)_n \qquad d \geq 3$

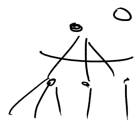
$U(G_n) \longrightarrow$ Canopy tree = random rooted tree



geo(p)   p well-chosen.

+ G_n = ER random graph $G(n, \frac{d}{n})$ $V = \{1, \dots, n\}$ each edge is present iid with probability $\frac{d}{n} \wedge 1$

a.s. $U(G_n) \xrightarrow[n \to \infty]{} GW(Poi(d))$

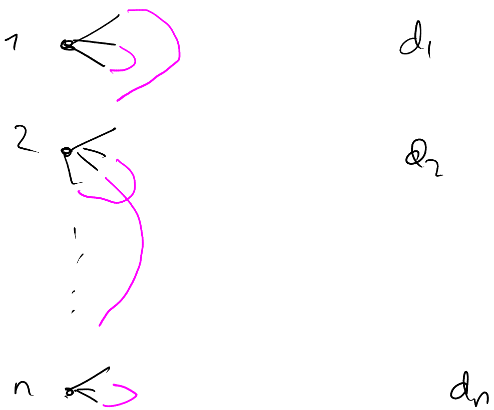


$Poi(d)$

+ More generally $d^n = (d_1^n, \dots, d_n^n)$ a sequence of integers $\frac{1}{n} \sum_{i=1}^{n} \delta_{d_i} \xrightarrow{w} P.$

such that $\sum d_i$ is even and $\underline{\sup_n \frac{1}{n} \sum (d_i^n)^2} < \infty$

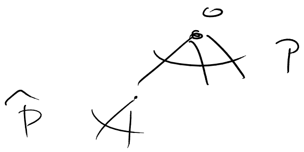
configuration model



uniformly sampled pairing / matching of the half-edges.

$G_n = CM((d_1, \dots, d_n))$ random multi-graph where $\deg(x) = d_x \quad \forall x \in \{1, \dots, n\}$

a.s. $U(G_n) \longrightarrow UGW(P)$



unimodular Galton-Watson tree with degree distribution P:

$$\hat{P}(k) = \frac{(k+1)\,P(k+1)}{\sum \ell\, P(\ell)}$$

Note that $\widehat{\mathrm{Poi}(c)} = \mathrm{Poi}$

see monograph by Remco van der Hofstadt.

<u>Unimodularity</u> Let $\tilde{\mathcal{G}}_{\bullet\bullet}$ be the set of unlabeled doubly rooted graphs (G, x, y)

$(\tilde{\mathcal{G}}_{\bullet\bullet}, d_{loc})$ is also a complete separable metric space

<u>Def</u> $\rho \in \mathcal{P}(\tilde{\mathcal{G}}_{\bullet})$ is unimodular if for all non-negative measurable functions $f : \tilde{\mathcal{G}}_{\bullet\bullet} \to \mathbb{R}_+$

Aldous-Lyons

$$\mathbb{E}_\rho \sum_{v \in V} f(G, o, v) = \mathbb{E}_\rho \sum_{v \in V} f(G, v, o)$$

"Mass Transport Principle" $f(G, x, y)$ = mass sent from x to y

<u>Lemma</u> If G is finite then $U(G)$ is unimodular

<u>proof</u> $\mathbb{E}_{U(G)} \sum_{v \in V} f(G, o, v) = \frac{1}{|V_G|} \sum_{u \in V_G} \sum_{v \in V_G(u)} f(G(u), u, v)$

$v \in V(u)_{G(u)} \iff u \in V(v)_{G(v)}$

$$= \frac{1}{|V_G|} \sum_{v \in V_G} \sum_{u \in V(v)_G} f(G(v), u, v)$$

$$= \mathbb{E}_{U(G)} \sum_{u \in V} f(G, u, o) .$$

Def $\rho \in \mathcal{P}(\tilde{\mathcal{G}}_\bullet)$ is sofic if there exists a finite sequence of graphs such $U(G_n) \to \rho$.

$\mathcal{S}_{sofic}(\tilde{\mathcal{G}}_\bullet) = \overline{\{U(G) : G \text{ finite}\}}$

Lemma: $\mathcal{S}_{uni}(\tilde{\mathcal{G}}_\bullet)$ the set of unimodular measures is closed for the weak topology. In particular $\mathcal{S}_{sofic}(\tilde{\mathcal{G}}_\bullet) \subseteq \mathcal{S}_{uni}(\tilde{\mathcal{G}}_\bullet)$.

The converse is open!