$G = K_n$ complete graph on n vertices

$\mathrm{perc}(K_n, p)$ = Erdős-Renyi random graph. $\mathbb{G}(n,p)$

Spectral measure of a graph

$G = (V, E)$ V = countable set
locally finite. undirected

Adjacency operator

$C_c(V) \subseteq \ell^2(V)$ finitely supported vectors

$x \in V$ $(A\psi)(x) = \sum_{e=\{x,y\} \in E} \psi(y) \in \ell^2(V)$

in matrix form $\langle \delta_x, A\delta_y \rangle = A_{xy}$ = multiplicity of $\{x,y\}$.

$A_{xy} = A_{yx}$ $\therefore A$ is symmetric

If $\deg(x) \le d$ $\forall x \in V$

$$\|A\psi\|_2^2 = \sum_{x \in V} \Big(\sum_{e=\{x,y\}} \psi(y) \Big)^2$$

$$\overset{CS}{\le} \sum_{x \in V} \deg(x) \sum_{e=\{x,y\}} \psi(y)^2$$

$$\le \sum_{y \in V} d^2 \, \psi(y)^2$$

$$= d^2 \, \|\psi\|_2^2.$$

$\|A\|_{2 \to 2} \le d.$

There are many others local operators including:

Laplacian: $L = D - A$ where $(D\psi)(x) = \deg(x)\,\psi(x)$

Combinatorial Laplacian $\Delta = D^{-1/2} A D^{-1/2}$ if $\forall x$ $\deg(x) \geq 1$

$P = D^{-1} A = D^{-1/2} \Delta D^{-1/2}$ transition kernel of the simple random walk on G.

If G is d-regular then $D = d \cdot 1_V$ and A commutes

Spectral measure at a vector

Assume first that G has bounded degree: $\sup_x \deg(x) = d < \infty$
then A is bounded and extend to a bounded symmetric operator on $\ell^2(V)$

Its spectrum $\sigma(A) = \{ z \in \mathbb{C} : A - z$ has a bounded inverse$\}^c$

$$\sigma(A) \subseteq [-d, d]$$

Th (Riesz-Markov-Kakutani representation theorem) If $L : \mathbb{R}(x) \to \mathbb{R}$ is linear and non-negative (if $P(x) \geq 0$ $\forall x$ then $L(P) \geq 0$) then there exists a (non-negative) measure μ such that $\forall P \in \mathbb{R}[x]$

$$L(P) = \int P(\lambda)\, d\mu(\lambda).$$

If $L(1) = 1$ then μ is a probability measure

Fact: If $m_{2k} = L(x^{2k}) \leq d^{2k}$ then μ is unique and has support $\subseteq (-d, d)$
(exercise)

Carleman's condition: $\sum_{k=1}^{\infty} m_{2k}^{-1/2k} = \infty$ then μ is unique

In part. if $\phi \in C_c(V)$ $L(P) = \langle \phi, P(A)\phi \rangle$.

is linear and non-negative $P = Q^2$ $L(P) = \langle \phi, Q^2(A)\phi \rangle$

$Q(A)^* = Q(A)$ $= \langle Q(A)\phi, Q(A)\phi \rangle$

$= \| Q(A)\phi \|_2^2 \geq 0$

Hence if $\|\phi\|_2 = 1$ then there exists a probability measure =

spectral measure at vector ϕ such that:

$$\boxed{\langle \phi, P(A)\phi \rangle = \int P(\lambda)\, d\mu_G^{\phi}(\lambda)}$$

If $\|A\| \leq d$ then μ_G^{ϕ} is unique and has support $\subseteq (-d, d)$.

$x \in V$ $\langle \delta_x, A^k \delta_x \rangle = \int \lambda^k \, d\mu_G^{\delta_x}(\lambda) =$ no of closed walks $\gamma = (\gamma_0, \dots \gamma_k)$
with $\gamma_0 = \gamma_k = x$ and $\{\gamma_t, \gamma_{t+1}\} \in E$
$t = 0, \dots k-1$

In terms of functional calculus $f(A)$ f bounded measurable fct.

$E \subseteq \mathcal{B}(\mathbb{R})$ $\quad \mu_G^\phi(E) = \langle \phi, \mathbb{1}_E(A)\phi \rangle.$

$$\Pi : \begin{cases} E \mapsto \mathbb{1}_E(A) \\ \mathcal{B}(\mathbb{R}) \to \text{projections on } \ell^2(V) \end{cases}$$

is called the resolution of identity of A

"$A = \int \lambda \, d\Pi(\lambda)$" Reed-Simon or Kadison-Ringrose.

Ex G is finite $V = \{1, \dots, n\}$. $A = \sum_{i=1}^n \lambda_i u_i u_i^*$

$$\Pi = \sum_{i=1}^n \delta_{\lambda_i} \cdot u_i u_i^*$$

$$\boxed{\mu_G^\phi = \sum_{i=1}^n \delta_{\lambda_i} \, |\langle u_i, \phi \rangle|^2.}$$

Resolvent operator $R(z) = (A - z)^{-1}$ $\quad z \notin \sigma(A)$

$$\langle \phi, R(z)\phi \rangle = \int \frac{d\mu_G^\phi(\lambda)}{\lambda - z}$$ is the Cauchy-Stieltjes transform of μ_G^ϕ.

Everything extends in the unbounded case (i.e. $\sup_{x \in V} \deg(x) = \infty$) provided that A is essentially self-adjoint. $\bar{A} = A^*$ $\quad C_c(V) \subseteq D(\bar{A}) \subseteq \ell^2(V)$

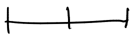
L is essentially self-adjoint if G is locally finite.

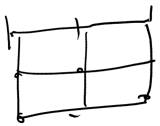
Operations on graphs and spectrum

① Cartesian product of graphs

$G_i = (V_i, E_i)$ $\quad i=1,2$ $\qquad G = G_1 \times G_2 = (V, E)$

$V = V_1 \times V_2$ $\qquad \{(x_1,x_2),(y_1,y_2)\} \in E$ iff either $x_1 = y_1$ and $\{x_2,y_2\} \in E_2$
or $x_2 = y_2$ and $\{x_1,y_1\} \in E_1$


$G_1 = G_2$


$G_1 \times G_2$

$$A_{G_1\times G_2} = A_{G_1} \otimes I_{V_2} + I_{V_1} \otimes A_{G_2}$$

Lemma

$$\mu^{\delta_{(x_1,x_2)}}_{G_1\times G_2} = \mu^{\delta_{x_1}}_{G_1} * \mu^{\delta_{x_2}}_{G_2}$$

if A_{G_1} and A_{G_2} essentially self-adjoint.

convolution of probability measures.

Proof

$$\int \lambda^k \, d\mu^{\delta_{x_1,x_2}}_{G_1\times G_2}$$

= Nb of closed walks in $G_1 \times G_2$ of length k starting at (x_1,x_2) $= W_k^{G_1\times G_2}(x_1,x_2)$

$$= \sum_{\ell=0}^{k} \underbrace{W^{G_1\times G_2}_{\ell,k}}(x_1,x_2)$$

closed walks with ℓ steps in G_1
$k-\ell$ —— G_2

$$W^{G_1\times G_2}_{e,k}(x_1,x_2) = \binom{e}{k} W^{G_1}_{e}(x_1)\, W^{G_2}_{k-e}(x_2)$$

$$\int \lambda^k \, d\mu^{\delta_{(x_1,x_2)}}_{G_1\times G_2} = \sum_{e=0}^{k} \binom{e}{k} \int \lambda^e \, d\mu^{\delta_x}_{G_1} \cdot \int \lambda^{k-e} \, d\mu^{\delta_{x_2}}_{G_2}$$

$$= \int \lambda^k \, d\mu^{\delta_{x_1}}_{G_1} * \mu^{\delta_{x_2}}_{G_2}(\lambda).$$

② Tensor product of graphs

$G_1\otimes G_2 = (V,E)$ $\quad V = V_1\times V_2$ $\quad \{(x_1,x_2),(y_1,y_2)\}\in E$ if $(x_1,y_1)\in E_1$ and $(x_2,y_2)\in E_2$

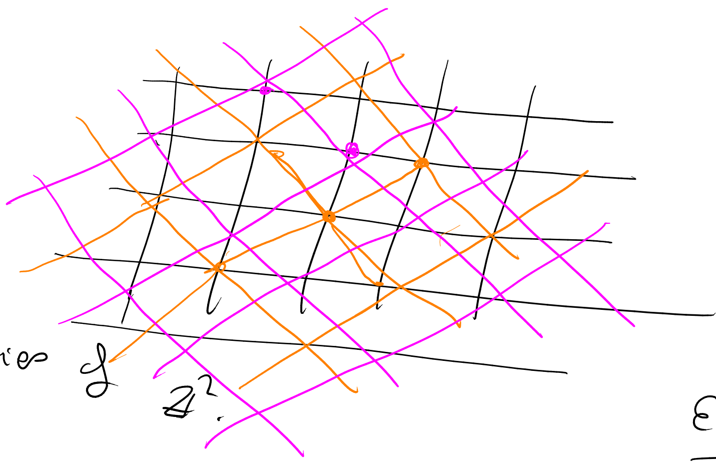
$$A_{G_1\otimes G_2} = A_{G_1}\otimes A_{G_2}$$

Lemma $\quad \mu^{\delta_{(x_1,x_2)}}_{G_1\otimes G_2} = \mu^{\delta_{x_1}}_{G_1}\odot \mu^{\delta_{x_2}}_{G_2}$ where $X\sim\mu$ and $Y\sim\nu$

$X \perp\!\!\!\perp Y$ independent

$\mu\odot\nu$ is the law of XY.

$\mathbb{Z}\otimes\mathbb{Z}$ is isomorphic to two copies of $\mathbb{Z}^2$.



if $\nu = \mu^{\delta_0}_{\mathbb{Z}}$

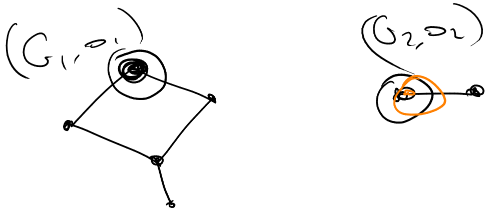
$$\boxed{\nu * \nu = \nu\odot\nu} \quad (*)$$

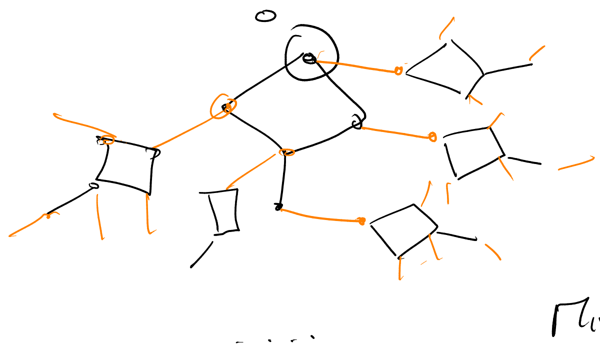
Exercise: $(*) \Rightarrow \nu(dx) = \mathbb{1}_{|x|\le 2}\frac{1}{\pi\sqrt{4-x^2}}\,dx$

③ Free product of graphs (rooted)

Def rooted graph (G,o) : is a connected graph with a distinguished vertex $o \in V$ called the root

$(G_1,o_1) * (G_2,o_2)$ free product





Lemma

$$\mu^{\delta_{o_1}}_{G_1 * G_2} = \mu^{\delta_{o_1}}_{G_1} \oplus \mu^{\delta_{o_2}}_{G_2}$$

Mingo–Speicher

Garza-Vargas & Kulkarni (2021)

If Γ_1 and Γ_2 are finitely generated groups and $\Gamma = \Gamma_1 * \Gamma_2$ free product

S_i generating set symmetric $\quad G_i = \mathrm{Cay}(\Gamma_i, S_i)$

$$(G_1,o_1) * (G_2,o_2) = \mathrm{Cay}(\Gamma_1 * \Gamma_2, S_1 \sqcup S_2)$$

(average)

Spectral measure of finite graphs

$G=(V,E)$ $\quad |V|=n$

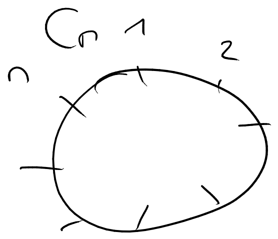
define $\boxed{\mu_G = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i}}$ where $(\lambda_1,\dots,\lambda_n)$ are the eigenvalues of A counting multiplicities.

empirical distribution of eigenvalues / density of states.

We have $\boxed{\mu_G = \frac{1}{n}\sum_{x\in V} \mu_G^{\delta_x}}$ spatial average of spectral measures at vertices

indeed
$$\sum_{x\in V}\mu_G^{\delta_x} = \sum_{x\in V}\sum_i \delta_{\lambda_i}\cdot\langle \delta_x, u_i\rangle^2$$
$$= \sum_i \delta_{\lambda_i}$$

Cycles



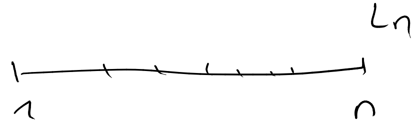
$$A = S + S^* \qquad S = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \text{ permutation}$$

$$\lambda_k(S) = e^{i\frac{2\pi k}{n}} \qquad k=0,\dots,n-1.$$

hence $\lambda_k(A) = 2\cos\left(\frac{2\pi k}{n}\right)$ $\quad k=0,\dots,n-1$ $\qquad U$ unif $(0,1)$

$$\mu_{C_n} = \frac{1}{n}\sum \delta_{2\cos\left(\frac{2\pi k}{n}\right)} \xrightarrow[n\to\infty]{\text{weakly}} \nu = \text{arcsine law with} = \text{law of } 2\cos(2\pi U).$$

Line segment



$$\lambda_k = 2\cos\left(2\pi \frac{k}{n+1}\right) \qquad 1 \le k \le n$$

$$\mu_{L_n} \xrightarrow[n\to\infty]{} \nu$$

Spectral measures of Cayley graphs

Γ countable group $\quad S = S^{-1}$ finite generating set

$G = \mathrm{Cay}(\Gamma, S) \qquad o$ unit of Γ

$$A = \sum_{g \in S} \lambda(g) \qquad B(\ell^2(\Gamma))$$

$g, h \in \Gamma$

$\lambda(g)$ left-regular representation: $\lambda(g)(\delta_h) = \delta_{gh}$.

A is an element of the (left) group algebra generated by $\lambda(g), g \in \Gamma$.

This group algebra generates a von Neumann algebra $\mathcal{M}$

$T \in \mathcal{M} \qquad \tau(T) = \langle \delta_o, T\delta_o \rangle \qquad$ tracial state

$\tau(T^*) = \overline{\tau(T)}, \qquad \tau(TT^*) \ge 0 \quad = 0$ iff $T = 0$

$\tau(AB) = \tau(BA)$, normal $\quad T_\alpha$ increasing net $\to T \quad \tau(T_\alpha) \to \tau(T)$

$$\boxed{\mu_G := \mu_G^{\delta_x}} \qquad x \in \Gamma \qquad \text{Plancherel measure}$$

$$\int \lambda^k \, d\mu_G(\lambda) = \tau(A^k).$$