SPECTRUM OF RANDOM GRAPHS

Brief introduction

+ understand rich connections between geometry of graphs and their spectra. With a focus on large finite random graphs

+ many motivations: theoretical computer science / statistical disordered media / geometric group theory / probability / data sciences...

+ rich topic in growing development with contributions: probability / operator algebras / mathematical physics / algebraic topology / ...

Divided into two complementary parts:

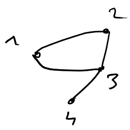
Part I: inside the spectrum: far from the edge

Part II: at the edge of spectrum: extremal eigenvalues and eigenvectors.

PRELIMINARY: what is a graph?

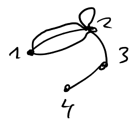
Def 1 $G=(V,E)$ $\quad$ V countable set, vertices

E subset of pairs of vertices (unordered)



$V=\{1,2,3,4\}$ $\quad$ $E=\{\{1,2\},\{2,3\},\dots\}$

unoriented simple graph

Def 2  multi-edges, loops $\quad$ E = multiset: repetitions are allowed

$E=\{\{1,2\},\{1,2\},\{1,2\},\{2\},\dots\}$ $\quad$ multi-graph.

$\deg(1)=3$

$\deg(2)=6$ $\quad$ edge multiplicity: $m_{\{1,2\}}=3$

Def (weighted graphs) simple graph $G=(V,E)$ $\quad$ $w_E: E\to\Omega$ $\quad$ $w_V: V\to\Omega$

The degree of a vertex $x\in V$ $\quad$ $\deg_G(x)$ = nb of adjacent edges

Def a graph is locally finite if $\forall x\in V$ $\deg(x)<+\infty$

has bounded degrees if $\sup_{x\in V}\deg(x)<+\infty$

is d-regular if $\forall x\in V$ $\deg(x)=d$

Def $\mathrm{Aut}(G)$ is the set of bijections $\phi: V\to V$ such that $\phi(G)=G$ where $\phi(\{u,v\}):=\{\phi(u),\phi(v)\}$.

the graph is transitive: if $\forall x,y\in V$ there $\phi\in\mathrm{Aut}(G)$: $\phi(x)=y$. $(*)$

quasi-transitive: finitely many orbits for the equivalence relation $x\sim y$ if $(*)$ holds.

Ex (Cayley graphs) Γ a countable group with S a finite symmetric set of generators $x \in S \Leftrightarrow x^{-1} \in S$.

$\mathrm{Cay}(\Gamma, S)$ (left) Cayley graph $V = \Gamma$ and $e = \{x, y\} \in E$ if and only if $y = sx$ for some $s \in S$
$yx^{-1} \in S$

Lemma: $\mathrm{Cay}(\Gamma, S)$ is transitive

Ex (Percolation) $G = (V, E)$ a fixed graph $p \in [0,1]$ $\mathrm{perc}(G, p) = (V, E_p(\omega))$

where $e \in E_p(\omega)$ independently with proba p (open) <u>bond</u> percolation

a given vertex is open ——— and $e \in E_p(\omega)$ if both adjacent vertices are open <u>site</u> percolation