Exponential Growth in the rational homology of free loop spaces

X - 1-connected

LX - free loop space $= \mathrm{Map}(S^1, X)$.

$\exists$ fibration $\Omega X \to LX \longrightarrow X$

Goal: Study growth in $H_*(LX;\mathbb{Q})$.

Conjecture: (Gromov - Vigué-Poirrier) If X is a rationally hyperbolic finite CW-complex then $H_*(LX;\mathbb{Q})$ has exponential growth.

Known - for wedge of spheres (Vigué-Poirrier)

- for $M \# N$ where M, N are closed connected manifolds, connected sum is not trivial (Lambrechts)

- X is coformal (Lambrechts).

Félix-Halperin-Thomas studied growth using Sullivan algebras.

Def: Let $V = \{V_i\}_{i \geq 0}$ be a graded vector space of finite type and let

$$r_k = \sum_{i \leq k} \dim V_i .$$

We say V grows exponentially if $\exists$ constants

$1 < c_1 < c_2 < \infty$ such that for some M

$$c_1^k < r_k < c_2^k \quad \forall k \geq M.$$

Def: The log index of V is

$$\text{log index}(V) = \limsup_i \frac{\log(\dim V_i)}{i}.$$

Remarks: The log index comes from the Hilbert Series

$$V(z) = \sum_{i=1}^{\infty} \dim V_i \, z^i$$

This has radius of convergence $\rho_V = e^{-\text{log index}(V)}$.

V grow exponentially if $0 < \rho_V < 1 \Rightarrow \log\text{index}(V) > 0$.

For spaces,

$$\log\text{index}(\pi_*(X)) = \log\text{index}(\pi_{\geq 2}(X) \otimes \mathbb{Q}).$$

A more precise version of exponential growth is:

<u>Def</u>: A graded vector space $V = \{V_i\}_{i \geq 0}$ of finite type has <u>controlled exponential growth</u> if $0 < \log\text{index}(V) < \infty$ and for each $\lambda > 1$ there is an infinite sequence $n_1 < n_2 < \cdots$ such that
$$n_{i+1} < \lambda n_i \quad \forall i \geq 0 \quad \text{and}$$

$$\dim V_{n_i} = e^{\alpha_i n_i} \quad \text{where} \quad \alpha_i \to \log\text{index}(V).$$

<u>Thm</u> (Félix-Halperin-Thomas) If X is 1-connected with $H_*(X;\mathbb{Q})$ is finite type then

$$\begin{aligned}\log\text{index}\, H_*(\Omega X;\mathbb{Q}) &\leq \log\text{index}(\pi_*(X)) \\ &= \log\text{index}(H_*(\Omega X);\mathbb{Q}).\end{aligned}$$

Def: Let X be a 1-connected space with $H_*(X;\mathbb{Q})$ of finite type and

$$0 < \log \text{index } H_*(\Omega X;\mathbb{Q}) < \infty.$$

Then LX has <u>good exponential growth</u> if $H_*(LX;\mathbb{Q})$ has controlled exponential growth and

$$\log\text{index } H_*(LX;\mathbb{Q}) = \log\text{index } H_*(\Omega X;\mathbb{Q}).$$

<u>Thm</u>: (Félix-Halperin-Thomas) If X is a 1-connected finite type wedge of spheres then LX has good exponential growth.

<u>Thm</u>: (Félix-Halperin Thomas) Let $F \to Y \to Z$ be a fibration where all spaces are 1-connected and have rational homology of finite type. If

$$\log\text{index }(\pi_*(Z)) < \log\text{index }(\pi_*(Y))$$

then LY has good exponential growth iff LF does.

<u>Ex</u>: If Z is rationally elliptic then $\log\text{index }(\pi_*(Z)) = 0$. So if

Y is rationally hyperbolic then $\pi_*(Y)$ grows exponentially $\Rightarrow$ log index $(\pi_*(Y)) > 0$

$\Rightarrow$ $\mathcal{L}Y$ has good exponential growth iff $\mathcal{L}F$ does.

Recall: In a htpy cofibration $\Sigma A \xrightarrow{f} Y \xrightarrow{h} Z$ the map f is <u>inert</u> if Ωh has a right htpy inverse.

<u>Def</u>: f is <u>strongly inert</u> if it is inert and

$$\text{log index}(\pi_*(Z)) < \text{log index}(\pi_*(Y)).$$

<u>Thm</u>: (Félix-Halperin-Thomas) (paraphrased)

Suppose $\exists$ htpy cofibration

$$S^n \xrightarrow{f} Y \xrightarrow{h} Z.$$

where f is strongly inert. Then $\mathcal{L}Y$ has good exponential growth.

- Pf uses Sullivan models.

We'll give a different, simpler proof using integral! decomposition methods and will be more general.

<u>Thm</u> (Huang–T) Suppose $\exists$ htpy cofibration

$$\Sigma A \xrightarrow{f} Y \xrightarrow{h} Z$$ where A, Z are not rationally contractible and f is strongly inert. Then ΩY has good exponential growth.

<u>Pf</u>: f inert $\Rightarrow$ Ωh has a right htpy inverse

$\Rightarrow$ by Cor C, $\exists$ htpy fibration

$$\Omega Z \ltimes \Sigma A \longrightarrow Y \xrightarrow{h} Z$$

and there is a htpy equivalence

$$\Omega Y \simeq \Omega Z \times \Omega(\Omega Z \ltimes \Sigma A).$$

f is strongly inert

$\Rightarrow$ log index $\pi_*(Z) <$ log index $\pi_*(Y)$

$\Rightarrow$ by FHT, ΩY has good exponential growth iff $\Omega Z \ltimes \Sigma A$ does.

Rationally, any suspension is htpy equivalent to a wedge of spheres.

Our case $$\Omega Z \ltimes \Sigma A \simeq (\Omega Z \wedge \Sigma A) \vee \Sigma A$$
$$\simeq \Sigma\big((\Omega Z \wedge A) \vee A\big).$$

$\Rightarrow$ $\Omega Z \ltimes \Sigma A$ is htpy equiv to a wedge of spheres.

$\Rightarrow$ By FHT, π(wedge of spheres) has good exponential growth.

$\Rightarrow$ $\Omega Z \ltimes \Sigma A$ has good exponential growth (noting A, Z not rationally contractible so $\Omega Z \ltimes \Sigma A$ is a wedge of at least 2 spheres). $\square$

An application to PD complexes.

<u>Thm</u> (Huang–T) Let M be a 1-connected n-dim PD-complex satisfying htpy cofibrations

$$S^{n-1} \longrightarrow S^m \vee S^{n-m} \vee \Sigma J \longrightarrow M$$

$$\Sigma J \longrightarrow M \longrightarrow Q$$

where $H^*(Q) \cong H^*(S^m \times S^{n-m})$. Then LM has good exponential growth.

$\Rightarrow$ The Gromov - Vigué - Poirrier conjecture holds.

Pf: Consider the htpy cofibration

$$\Sigma J \xrightarrow{f'} M \xrightarrow{h'} Q$$

- We saw $\Omega h'$ has a right htpy inverse
  $\Rightarrow f'$ is inert.

- We saw $\exists$ htpy fibration

$$\Omega Q \ltimes \Sigma J \to M \xrightarrow{h'} Q$$

- We saw $\Omega Q \simeq \Omega S^m \times \Omega S^{n-m}$
  $\Rightarrow Q$ is elliptic
  $\Rightarrow$ log index $\pi_*(Q) = 0$.

- $$\Omega Q \ltimes \Sigma J \simeq (\Omega S^m \times \Omega S^{n-m}) \ltimes \Sigma J$$
$$\simeq (\Omega S^m \times \Omega S^{n-m}) \wedge \Sigma J \vee \Sigma J$$
  - wedge of spheres

rationally.

$\Rightarrow$ logindex $\pi_*(\Omega Q \rtimes \Sigma J) > 0$.

- We saw $\Omega M \simeq \Omega Q \times \Omega(\Omega Q \rtimes \Sigma J)$

$\Rightarrow$ logindex $\pi_*(M) > 0$.

$\Rightarrow$ f' is strongly inert.

Hence, by the previous theorem, hM has good exponential growth. $\square$