The Homology of Loops on PD-complexes.

- $M \in \mathcal{M}_n$.
- Goal is to calculate $H_*(\Omega M)$ as an algebra.
- Generically $\Omega M \simeq \prod X_i$

$$\text{then} \quad H_*(\Omega M) \underset{\substack{\uparrow \\ \text{as modules.}}}{\cong} \bigotimes H_*(X_i)$$

<u>**Bott-Samelson Thm**</u>: If X is a path-connected space then

$$H_*(\Omega \Sigma X) \underset{\substack{\uparrow \\ \text{algebra}}}{\cong} T(\widetilde{H}_*(X)) \qquad \text{(field coeffs)}$$

and the suspension $X \xrightarrow{E} \Omega \Sigma X$ induces the inclusion $\widetilde{H}_*(X) \hookrightarrow T(\widetilde{H}_*(X))$.

<u>More</u>: If X is a suspension then this isomorphism is as Hopf algebras.

<u>Thm</u>: Suppose $\exists$ htpy cofibration $\Sigma A \xrightarrow{f} \Sigma Y \xrightarrow{h} Y'$

where Ωh has a right htpy inverse.
Let $\tilde{f} : A \to \Omega\Sigma Y$ be the adjoint of f.
Let $R = \mathrm{Im}(\tilde{f}_*)$. Then $\exists$ algebra isomorphism

$$H_*(\Omega Y') \cong T(\tilde{H}_*(Y)) / (R)$$

where (R) is the 2-sided ideal generated by R.

<u>Remark</u>: If Y is a suspension then this iso is as Hopf algebras.

<u>Pf</u>: ① $\Omega\Sigma Y \xrightarrow{\Omega h} \Omega Y'$

$$\Rightarrow \quad T(\tilde{H}_*(Y)) \xrightarrow{(\Omega h)_*} H_*(\Omega Y') \quad \text{- algebra map}$$

As Ωh has a right htpy inverse, get $(\Omega h)_*$ is an epimorphism.

② The adjoint $\hat{f} : A \to \Omega\Sigma Y$ is htpic to the composite

$$A \xrightarrow{E} \Omega\Sigma A \xrightarrow{\Omega f} \Omega\Sigma Y.$$

Consider $\Omega\Sigma A$ $\Rightarrow \Omega h' \circ \hat{f} \simeq *$.

$$\begin{tikzcd} & \cdots \arrow[d, "\Omega f"] \arrow[dd, bend left=60, "*"] \\ A \arrow[ur] \arrow[r, "\tilde{f}"] \arrow[dr, bend right, "\text{null htpic}"'] & \Omega\Sigma Y \arrow[d, "\Omega h"] \\ & \Omega Y' \end{tikzcd}$$

As $R = \operatorname{Im} \tilde{f}_*$, and $(\Omega h)_*$ is an algebra map, get a factorization

$$\begin{tikzcd} T(\tilde{H}_*(Y)) \arrow[r, "(\Omega h)_*"] \arrow[d, "a"] & H_*(\Omega Y') \\ T(\tilde{H}_*(Y))/(R) \arrow[ur, dashed, "b"'] & \end{tikzcd}$$

where b is an algebra map and an epimorphism.

Claim: b is an isomorphism.

③ Now aim for a factorization

$$\begin{tikzcd} T(\tilde{H}_*(Y)) \arrow[r, "(\Omega h)_*"] \arrow[d, "a"] & H_*(\Omega Y') \arrow[dl, dashed, "c"] \\ T(\tilde{H}_*(X))/(R) & \end{tikzcd}$$

.

The htpy cofibration $\Sigma A \xrightarrow{f} \Sigma Y \xrightarrow{h} Y'$ has Ωh with a right htpy inverse so by the enhanced version of Gor C gives a htpy fibration

$$\Omega Y' \ltimes \Sigma A \xrightarrow{[\gamma, f]+f} \Sigma Y \xrightarrow{h} Y'$$

where $\gamma : \Sigma\Omega Y' \xrightarrow{\Sigma s} \Sigma\Omega\Sigma Y \xrightarrow{ev} \Sigma Y$

& s is a right htpy inverse for Ωh.

Think of

$$\Omega((\Omega Y' \wedge \Sigma A) \vee \Sigma A) \xrightarrow[\chi]{} \Omega\Sigma Y \xrightarrow{\Omega h} \Omega Y'$$

$\chi = \Omega([\gamma, f]+f)$

- splits $\Omega\Sigma Y \simeq \Omega Y' \times \Omega((\Omega Y' \wedge \Sigma A) \vee \Sigma A)$

$\Rightarrow \quad T(\tilde{H}_*(Y)) \cong H_*(\Omega Y') \otimes H_*(\Omega(\Omega Y' \wedge \Sigma A) \vee \Sigma A)$

iso as right $H_*(\Omega(\Omega Y' \wedge \Sigma A) \vee \Sigma A))$-modules

Consider

$$H_*(\Omega((\Omega Y' \wedge \Sigma A) \vee \Sigma A)) \xrightarrow{\chi_*} T(\tilde{H}_*(Y))$$

$\quad | \, a$

$$\downarrow$$

$$T(\tilde{H}_*(Y))/(R).$$

- X_* is an alg map so it is determined by what happens when precomposed with

$$(\Omega Y' \wedge A) \vee A \xrightarrow{E} \Omega((\Omega Y' \wedge \Sigma A) \vee \Sigma A)$$

- $X \circ E|_A = \tilde{f}$ and $R = \mathrm{Im}\, \hat{f}_*$

$$\Rightarrow a \circ X_* \circ E_*|_A = 0.$$

- $X \circ E|_{\Omega Y \wedge A}$ is a Samelson product $\langle \gamma', \tilde{f} \rangle$ for some map γ'.

$$\Rightarrow a \circ \langle \gamma', \hat{f} \rangle_* \underset{\substack{a \text{ is an} \\ \text{alg map}}}{=} \langle a \circ \gamma'_*, a \circ \hat{f}_* \rangle$$

$$= 0 \quad \text{as } a \circ \hat{f}_* = 0.$$

Therefore $a \circ X_* = 0$.

Thus $\exists$ factorization

$$\ldots$$

$$\begin{tikzcd}
T(\hat{H}_*(Y)) \arrow[r, "(\Omega h')_*"] \arrow[d, "a"] & H_*(\Omega Y') \arrow[dl, dashed, "c"] \\
T(\hat{H}_*(Y))/(R) &
\end{tikzcd}$$

where c is an algebra map, and an epimorphism.

(4) Compose:

$$T(\hat{H}_*(Y))/(R) \xrightarrow{c} H_*(\Omega Y') \xrightarrow{b} T(\hat{H}_*(Y))/(R) \xrightarrow{c} H_*(\Omega Y')$$

- c, b are both algebra maps, both epimorphisms.

- $b \circ c$, $c \circ b$ are alg maps, epimorphisms.

- $b \circ c$ is a self-map of $T(\hat{H}_*(Y))/(R)$

  $c \circ b$ " $H_*(\Omega Y')$

- so $c \circ b$, $b \circ c$ are self-maps of finite type modules that are epimorphisms $\Rightarrow$ they are isomorphisms.

- Also algebra maps $\Rightarrow$ alg iso's.

So $$H_*(\Omega Y') \underset{\text{alg}}{\cong} T(\tilde{H}_*(Y))/(R). \qquad \square$$

<u>Ex</u>! Suppose $M \in \mathcal{M}_n$ so $\exists$ htpy cofibrations

$$S^{n-1} \xrightarrow{f} S^m \vee S^{n-m} \vee J \xrightarrow{j} M$$

$$J \longrightarrow M \longrightarrow Q$$

where $H^*(Q) \cong H^*(S^m \times S^{n-m})$.

If $J = \Sigma \bar{J}$ then as Ωj has a right htpy inverse get an algebra iso

$$H_*(\Omega M) \cong T(\tilde{H}_*(S^{m-1} \vee S^{n-m-1} \vee \bar{J}))/(R)$$

where $R = \mathrm{Im}\, \hat{f}_*$.

Note: $\hat{f} : S^{n-2} \longrightarrow \Omega(S^m \vee S^{n-m} \vee \Sigma \bar{J})$

Let $r = \mathrm{Im}\, \hat{f}_*$ - r is just a single element in $T(\hat{H}_*(S^{m-1} \vee S^{n-m-1} \vee \overset{d}{\dots}))$

$\Rightarrow \quad H_*(\Omega M) \cong T(\hat{H}_*(S^{m-1} \vee S^{n-m-1} \vee \dots)) / (r)$.

— this is called one-relator algebras.

<u>Ex</u>: More concrete, a special case of $M \in \mathcal{M}_n$.

Let $M = \#_{i=1}^{d} (S^{m_i} \times S^{n-m_i})$ where $m_i \geq 2 \; \forall i$

M is a PD-complex, $M \in \mathcal{M}_n$.

$\exists$ htpy cofib

$$S^{n-1} \xrightarrow{f} \bigvee_{i=1}^{d} (S^{m_i} \vee S^{n-m_i}) \xrightarrow{j} M$$

where $f = \sum_{i=1}^{d} [\iota_{m_i}, \iota_{n-m_i}]$

sum of Whitehead product.

Adjoint: $\tilde{f} : S^{n-2} \to \Omega(\bigvee_{i=1}^{d} (S^{m_i} \vee S^{n-m_i}))$

$$H_*(S^{n-2}) \to T(\tilde{H}_*(\bigvee_{i=1}^{d} S^{m_i-1} \vee S^{n-m_i-1}))$$

$$\cong T(u_1,\dots,u_d,v_1,\dots,v_d)$$

$$|u_i| = m_i - 1, \quad |v_i| = n - m_i - 1$$

Then $\tilde{f}_*(\gamma_{n-2}) = [u_1,v_1] + \dots + [u_d,v_d]$.

Thus

$$H_*(\Omega(\# S^{m_i} \times S^{n-m_i}))$$

$$\underset{\text{algebra}}{\cong} T(u_1,\dots,u_d,v_1,\dots,v_d) \big/ \big([u_1,v_1] + \dots [u_d,v_d]\big)$$

If each $m_i, n-m_i \geq 3$, this is an iso as Hopf algebras.