Reminder: Let $A \xrightarrow{f} Y \xrightarrow{h} Z$ be a htpy cofibration with Ωh having a right htpy inverse then $\exists$ htpy fibration

$$\Omega Z \ltimes A \xrightarrow{\bar{f}} Y \xrightarrow{h} Z$$

and $\Omega Y \simeq \Omega Z \times \Omega(\Omega Z \ltimes A)$.

Nothing said about the htpy class of $\Omega Z \ltimes A \to Y$.

Goal: Find out more.

<u>Thm A</u>: Suppose $\exists$ htpy cofibration $A \to Y \to Y'$ and a map $Y' \xrightarrow{h'} Z$. Then there is a diagram of data

$$\begin{tikzcd}
 & E \arrow[r] \arrow[d] & E' \arrow[d] \\
A \arrow[r, "f"] & Y \arrow[r] \arrow[d, "h"] & Y' \arrow[d, "h'"] \\
 & Z \arrow[r, equal] & Z
\end{tikzcd}$$

- columns are htpy fibrations. If Ωh has a right htpy inverse then $\exists$ htpy cofibrations

$$\Omega Z \ltimes A \xrightarrow{\bar{\varphi}} E \longrightarrow E'$$

where $\bar{\varphi}$ is a quotient of

$$\Omega Z \times A \xrightarrow{1 \times g} \Omega Z \times E \xrightarrow{act} E$$

for an appropriate lift g of f.

More is true

<u>Thm</u>: In the case of a htpy cofibration $\Sigma A \xrightarrow{f} Y \longrightarrow Y'$ $\exists$ a htpy commutative diagram

$$\begin{tikzcd}
\Omega Z \ltimes \Sigma A \arrow[r, "\bar{\varphi}"] \arrow[d, "\simeq"] & E \arrow[d] \\
(\Omega Z \wedge \Sigma A) \vee \Sigma A \arrow[r, "{[\gamma, f] \vee f}"] & Y
\end{tikzcd}$$

where if $s : \Omega Z \longrightarrow \Omega Y$ is a right htpy inverse for Ωh then

$$\gamma : \Sigma \Omega Z \xrightarrow{\Sigma s} \Sigma \Omega Y \xrightarrow{ev} Y.$$

Cor: If there is a htpy cofibration $\Sigma A \xrightarrow{f} Y \xrightarrow{h} Z$ where Ωh has a right htpy inverse then $\exists$ htpy fibration

$$\Omega Z \ltimes \Sigma A \xrightarrow{[\gamma, f]+f} Y \xrightarrow{h} Z$$

Ex: Let $M \in \mathcal{M}_n$. So $\exists$ htpy cofibrations

$$S^{n-1} \xrightarrow{f} S^m \vee S^{n-m} \vee J \xrightarrow{j} M$$

$$J \to M \to Q$$

where $H^*(Q) \cong H^*(S^m \times S^{n-m})$.

Know Ωj has a right htpy inverse.

By Cor, get a htpy fibration

$$\Omega M \ltimes S^{n-1} \xrightarrow{[\gamma, f]+f} S^m \vee S^{n-m} \vee J \xrightarrow{j} M$$

and $\Omega(S^m \vee S^{n-m} \vee J) \simeq \Omega M \times \Omega(\Omega M \ltimes S^{n-1})$.

- a sort of Hilton-Milnor style Theorem.

Where do the Whitehead products come from?

Come from joining 2 seemingly distinct constructions.

(A) Suppose $\exists$ htpy fibration sequence

$$\Omega Z \xrightarrow{\partial} E \xrightarrow{f} Y \xrightarrow{h} Z$$

(Ωh is not assumed to have a right htpy inverse yet.)

$\exists$ htpy action $\quad \Omega Z \times E \xrightarrow{a} E \quad$ extending

$$\partial \perp 1 : \Omega Z \vee E \longrightarrow E$$

Observe

$$\theta : \Omega Y \times E \xrightarrow{\Omega h \times 1} \Omega Z \times E \xrightarrow{a} E$$

has $\theta|_{\Omega Y}$ is null htpic. So $\exists$ quotient map

$$\begin{tikzcd} \Omega Y \times E \arrow[r, "\theta"] \arrow[d] & E \\ {} & {} \arrow[u, "\simeq"'] \end{tikzcd}$$

$\Omega Y \ltimes E \qquad \theta$.

(B) Consider the htpy fibration diagram
(fib seq $\Omega Z \xrightarrow{\partial} E \xrightarrow{i} Y \xrightarrow{h} Z$)

$$\begin{tikzcd}
\Omega Y \ltimes E \arrow[r, "\Gamma"] \arrow[d] & E \arrow[d] \\
Y \vee E \arrow[r, "1 \perp p"] \arrow[d, "\text{pinch}"] & Y \arrow[d, "h"] \\
Y \arrow[r, "h"] & Z
\end{tikzcd}$$

Γ is an induced map of fibres.

<u>Thm</u> (Gray): $\bar{\theta}, \Gamma$ can be chosen to be homotopic.

Pf involves writing down an explicit htpy.

Next: better identify Γ.

Preliminary case: Ganea showed $\exists$ htpy fibration

$$\Omega X_1 * \Omega X_2 \xrightarrow{[ev_1, ev_2]} X_1 \vee X_2 \longrightarrow X_1 \times X_2$$

where

$$ev_1 : \Sigma\Omega X_1 \xrightarrow{ev} X_1 \hookrightarrow X_1 \vee X_2$$

$$ev_2 : \Sigma\Omega X_2 \xrightarrow{ev} X_2 \hookrightarrow X_1 \vee X_2.$$

Want to use this to identify the map from the fibre in

$$\Omega X_1 \ltimes X_2 \longrightarrow X_1 \vee X_2 \xrightarrow{\text{pinch}} X_1$$

Observe $\exists$ lift

$$\begin{tikzcd}
 & \Omega X_1 \ltimes X_2 \arrow[d] \\
\Omega X_1 * \Omega X_2 \arrow[r, "{[ev_1, ev_2]}"] \arrow[ur, dashed, "\lambda"] & X_1 \vee X_2 \arrow[d, "\text{pinch}"] \\
 & X_1
\end{tikzcd}$$

Suppose $X_2 = \Sigma \bar{X}_2$. Then get

$$\begin{tikzcd}
\Omega X_1 * \bar{X}_2 \arrow[d, "1 * E"] & & \\
\Omega X_1 * \Omega\Sigma\bar{X}_2 = \Omega X_1 * \Omega X_2 \arrow[rr, "\lambda"] & & \Omega X_1 \ltimes X_2 \arrow[d, phantom, "\shortparallel"] \\
 & & \Omega X_1 \ltimes \Sigma\bar{X}_2 \arrow[d, phantom, "\simeq"] \\
 & & (\Omega X_1 \wedge \Sigma\bar{X}_2) \vee \Sigma\bar{X}_2
\end{tikzcd}$$

a htpy equivalence

$$\begin{array}{c} \downarrow \text{pinch} \\ \Omega X_1 \wedge \Sigma \bar{X}_2 \end{array}$$

So get

$$\begin{tikzcd}
 & & \Omega X_1 \ltimes \Sigma \bar{X}_2 \simeq (\Omega X_1 \wedge \Sigma \bar{X}_2) \vee \Sigma \bar{X}_2 \arrow[d] \\
\Omega X_1 \wedge \Sigma \bar{X}_2 \arrow[r, "1 * E"] \arrow[rr, bend right, "{[ev_1, i_2]}"'] & \Omega X_1 * \Omega\Sigma \bar{X}_2 \arrow[ur, "\lambda"] \arrow[r, "{[ev_1, ev_2]}"] & X_1 \vee \Sigma \bar{X}_2
\end{tikzcd}$$

$i_2 : \Sigma \bar{X}_2 \hookrightarrow X_1 \vee \Sigma \bar{X}_2$

$\Rightarrow$ fibration

$$(\Omega X_1 \wedge \Sigma \bar{X}_2) \vee \Sigma \bar{X}_2 \xrightarrow{[ev_1, i_2] \perp i_2} X_1 \vee \Sigma \bar{X}_2 \xrightarrow{\text{pinch}} X_1$$

<u>Our case</u>:

Htpy cofibration $\quad \Sigma A \xrightarrow{f} Y \longrightarrow Y'$.

Lift

$$\begin{tikzcd}
 & F \arrow[d] \\
\Sigma A \arrow[ur, "g"] \arrow[r, "f"] & Y \arrow[d, "h"] \\
 & Z
\end{tikzcd}$$

Get:

$$\begin{tikzcd}
 & \Omega Y \ltimes \Sigma A \arrow[r, "1\ltimes g"] \arrow[d] & \Omega Y \ltimes E \arrow[r, "\Gamma"] \arrow[d] & E \arrow[d, "p"] \\
{[ev_1, i_2] + i_2} \arrow[ur] & Y \vee \Sigma A \arrow[r, "1\vee g"] \arrow[d, "\text{pinch}"] & Y \vee E \arrow[r, "1\vee p"] \arrow[d, "\text{pinch}"] & Y \arrow[d, "h"] \\
 & Y \arrow[r, equal] & Y \arrow[r, "h"] & Z
\end{tikzcd}$$

$$\Rightarrow \quad p \circ \Gamma \circ (1\ltimes g) \simeq [ev_1, f] + f$$

We have not yet used that Ωh has a right htpy inverse. Now we do.

Recall:

$$\begin{tikzcd}
\Omega Y \times E \arrow[r, "\Omega h\times 1"] \arrow[d] & \Omega Z \times E \arrow[r, "a"] & E \\
\Omega Y \ltimes E \arrow[urr, bend right, "\bar{\theta}"'] & &
\end{tikzcd}$$

Refine: Ωh has a right htpy inverse

$$\begin{tikzcd}
 & \Omega Y \times E \arrow[r] \arrow[d] & \Omega Z \times E \arrow[r, "a"] \arrow[d] & E \\
\Omega Z \ltimes E \arrow[r, "s\ltimes 1"] & \Omega Y \ltimes E \arrow[r] \arrow[urr, bend right=60, "\bar{\theta} \simeq \Gamma"'] & \Omega Z \ltimes E \arrow[ur, "\bar{a}"'] &
\end{tikzcd}$$

$$\Rightarrow \quad \Gamma \circ (s \rtimes 1) \simeq \bar{a}$$

So

$$\begin{aligned}
p \circ \Gamma \circ (1 \rtimes g) \circ (s \rtimes 1) &\simeq p \circ \Gamma \circ (s \rtimes 1) \circ (1 \rtimes g) \\
&\simeq p \circ \bar{a} \circ (1 \rtimes g) \\
&\simeq p \circ \bar{\varphi}.
\end{aligned}$$

$$p \circ \Gamma \circ (1 \rtimes g) \circ (s \rtimes 1) \simeq \big([ev_1, f] + f\big) \circ (s \rtimes 1) \simeq [\gamma, f] + f$$

$$\gamma : \Sigma\Omega Z \xrightarrow{\Sigma s} \Sigma\Omega Y \xrightarrow{ev} Y$$

ie -

$$\begin{tikzcd}
\Omega Z \rtimes \Sigma A \arrow[r, "\bar{\varphi}"] \arrow[d, "\simeq"] & E \arrow[r, "\text{cofib}"] \arrow[d, "p"] & E' \\
(\Omega Z \wedge \Sigma A) \vee \Sigma A \arrow[r, "{[\gamma, f] \perp f}"] & Y &
\end{tikzcd}$$