

Reminder: Let $A \xrightarrow{f} Y \xrightarrow{h} Z$ be a htpy cofibration with sh having a right htpy inverse then \exists htpy fibration

$$\Omega Z \times A \xrightarrow{\bar{f}} Y \xrightarrow{h} Z$$

and $\Omega Y \cong \Omega Z \times \Omega(\Omega Z \times A)$.

Nothing said about the htpy class of $\Omega Z \times A \rightarrow Y$.

Goal: Find out more.

Thm A: Suppose \exists htpy cofibration $A \rightarrow Y \rightarrow Y'$ and a map $Y' \xrightarrow{h'} Z$. Then there is a diagram of dots

$$\begin{array}{ccccc} & & E & \longrightarrow & E' \\ & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & Y & \longrightarrow & Y' \\ & & \downarrow h & & \downarrow h' \\ & & Z & = & Z \end{array}$$

- columns are htpy fibrations. If sh has a right htpy inverse then \exists htpy cofibrations

$$\Omega Z \times A \xrightarrow{\bar{f}} E \rightarrow E'$$

where \bar{f} is a quotient of

$$\Omega Z \times A \xrightarrow{1 \times g} \Omega Z \times E \xrightarrow{\text{act}} E$$

for an appropriate lift g of f .

More is true

Thm: In the case of a htpy cofibration
 $\Sigma A \xrightarrow{f} Y \rightarrow Y'$ \exists a htpy commutative
 diagram

$$\begin{array}{ccc} \Omega Z \times \Sigma A & \xrightarrow{\bar{f}} & E \\ \downarrow \cong & & \downarrow \\ (\Omega Z \wedge \Sigma A) \vee \Sigma A & \xrightarrow{[\gamma, f] \vee f} & Y \end{array}$$

where if $s: \Omega Z \rightarrow \Omega Y$ is a right
 htpy inverse for Σh then

$$\gamma: \Sigma \Omega Z \xrightarrow{\Sigma s} \Sigma \Omega Y \xrightarrow{\text{ev}} Y.$$

Cor: If there is a htpy cofibration
 $\Sigma A \xrightarrow{f} Y \xrightarrow{h} Z$ where Σh has a
 right htpy inverse then \exists htpy
 fibration

$$\Omega Z \times \Sigma A \xrightarrow{[\gamma, f] + f} Y \xrightarrow{h} Z$$

Ex: Let $M \in \mathcal{M}_n$. So \exists htpy cofibrations

$$S^{n-1} \xrightarrow{f} S^m \vee S^{n-m} \vee J \xrightarrow{j} M$$

$$J \rightarrow M \rightarrow Q$$

where $H^*(Q) \cong H^*(S^m \times S^{n-m})$.

Now Σj has a right htpy inverse

By Cor, get a htpy fibration

$$\Omega M \times S^{n-1} \xrightarrow{[\gamma, f] + f} S^m \vee S^{n-m} \vee J \xrightarrow{j} M$$

and $\Omega(S^m \vee S^{n-m} \vee J) \cong \Omega M \times \Omega(S^m \times S^{n-1})$.

- a sort of Hilton-Milnor style Theorem.

Where do the Whitehead products come from?

Come from joining 2 seemingly distinct constructions.

Ⓐ Suppose \exists htpy fibration sequence

$$\Omega Z \xrightarrow{\partial} E \xrightarrow{p} Y \xrightarrow{h} Z$$

(Ωh is not assumed to have a right htpy inverse yet.)

\exists htpy action $\Omega Z \times E \xrightarrow{a} E$ extending

$$\partial \perp 1 : \Omega Z \vee E \rightarrow E$$

Observe

$$\Theta : \Omega Y \times E \xrightarrow{\Omega h \times 1} \Omega Z \times E \xrightarrow{a} E$$

has $\Theta|_{\Omega Y}$ is null htpy. So \exists quotient map

$$\begin{array}{ccc} \Omega Y \times E & \xrightarrow{\Theta} & E \\ \downarrow & \nearrow & \cong \end{array}$$

$$\Omega Y \times E \xrightarrow{\theta}$$

ⓑ Consider the htpy fibration diagram
(fib seq $\Omega Z \xrightarrow{\rho} E \xrightarrow{\iota} Y \xrightarrow{h} Z$)

$$\begin{array}{ccc} \Omega Y \times E & \xrightarrow{\Gamma} & E \\ \downarrow & & \downarrow \\ Y \vee E & \xrightarrow{\iota \circ \rho} & Y \\ \downarrow \text{pinch} & & \downarrow h \\ Y & \xrightarrow{h} & Z \end{array}$$

Γ is an induced map of fibres.

Thm (Gray): $\bar{\theta}, \Gamma$ can be chosen to be homotopic.

pf involves writing down an explicit htpy.

Next: better identify Γ .

Preliminary case: Ganea showed \exists htpy fibration

$$\Omega X_1 * \Omega X_2 \xrightarrow{[\text{ev}_1, \text{ev}_2]} X_1 \vee X_2 \rightarrow X_1 \times X_2$$

where

$$ev_1: \Sigma \Omega X_1 \xrightarrow{ev} X_1 \hookrightarrow X_1 \vee X_2$$

$$ev_2: \Sigma \Omega X_2 \xrightarrow{ev} X_2 \hookrightarrow X_1 \vee X_2.$$

Want to use this to identify the map from the fibre in

$$\Omega X_1 * X_2 \longrightarrow X_1 \vee X_2 \xrightarrow{p_{\text{incl}}} X_1$$

Observe \exists lift

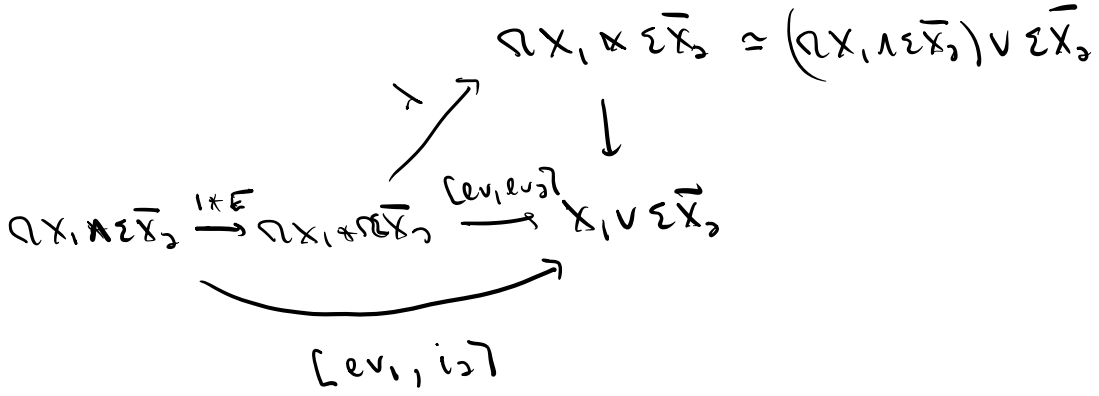
$$\begin{array}{ccc} & \lambda \rightarrow & \Omega X_1 * X_2 \\ & \swarrow & \downarrow \\ \Omega X_1 * \Omega X_2 & \xrightarrow{[ev_1, ev_2]} & X_1 \vee X_2 \\ & & \downarrow p_{\text{incl}} \\ & & X_1 \end{array}$$

Suppose $X_2 = \Sigma \bar{X}_2$. Then get

$$\begin{array}{ccc} \Omega X_1 * \bar{X}_2 & & \\ \downarrow 1 * E & \text{a homotopy equivalence} & \\ \Omega X_1 * \Omega \Sigma \bar{X}_2 = \Omega X_1 * \Omega X_2 & \xrightarrow{\lambda} & \Omega X_1 * X_2 \\ & & \parallel \\ & & \Omega X_1 * \Sigma \bar{X}_2 \\ & & \cong \\ & & (\Omega X_1 * \Sigma \bar{X}_2) \vee \Sigma \bar{X}_2 \end{array}$$

$\downarrow p_{in}$
 $\Omega X_1 \vee \Sigma \bar{X}_2$ ✓

So get



$i_2: \Sigma \bar{X}_2 \hookrightarrow X_1 \vee \Sigma \bar{X}_2$

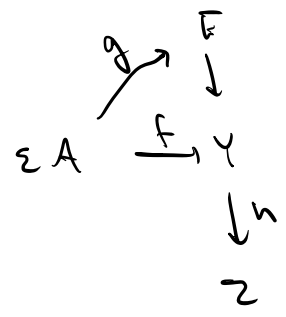
\Rightarrow fibration

$$(\Omega X_1 \vee \Sigma \bar{X}_2) \vee \Sigma \bar{X}_2 \xrightarrow{[lev_1, i_2] \circ i_2} X_1 \vee \Sigma \bar{X}_2 \xrightarrow{p_{in}} X_1$$

Our case:

Hopf cofibration $\Sigma A \xrightarrow{F} Y \rightarrow Y'$

Lift



Get:

$$\begin{array}{ccccc}
 \Omega Y \times \Sigma A & \xrightarrow{1 \times g} & \Omega Y \times E & \xrightarrow{\Gamma} & E \\
 \downarrow & & \downarrow & & \downarrow P \\
 \text{[Lev}_{1, i_2}] + i_2 & \nearrow & Y \vee \Sigma A & \xrightarrow{1 \vee g} & Y \vee E & \xrightarrow{1 \vee P} & Y \\
 \downarrow p_{\text{in}h} & & \downarrow p_{\text{in}h} & & \downarrow h & & \downarrow h \\
 Y & \xrightarrow{\quad\quad\quad} & Y & \xrightarrow{h} & Z
 \end{array}$$

$$\Rightarrow p \circ \Gamma \circ (1 \times g) \cong [\text{Lev}_{1, f}] + f$$

We have not yet used that Ωh has a right h - Ω inverse. Now we do.

Recall:

$$\begin{array}{ccc}
 \Omega Y \times E & \xrightarrow{\Omega h} & \Omega Z \times E \xrightarrow{a} E \\
 \downarrow & & \nearrow \bar{\theta} \\
 \Omega Y \times E & &
 \end{array}$$

Refine: Ωh has a right h - Ω inverse

$$\begin{array}{ccccc}
 \Omega Y \times E & \rightarrow & \Omega Z \times E & \xrightarrow{a} & E \\
 \downarrow & & \downarrow & \nearrow a & \uparrow \\
 \Omega Z \times E \xrightarrow{s \times 1} & \Omega Y \times E & \rightarrow & \Omega Z \times E & \\
 & & & & \bar{\theta} \cong \Gamma
 \end{array}$$

$$\Rightarrow \Gamma \circ (s \times 1) \simeq \bar{\alpha}$$

$$\text{So } p \circ \Gamma \circ (1 \times g) \circ (s \times 1) \simeq p \circ \Gamma \circ (s \times 1) \circ (1 \times g)$$

$$\cong p \circ \bar{\alpha} \circ (1 \times g)$$

$$\cong p \circ \tilde{\gamma}$$

$$\begin{array}{c} \cong \\ \text{is} \\ [\gamma, f] + f \end{array}$$

$$\gamma: \Sigma \Omega Z \xrightarrow{\Sigma s} \Sigma \Omega Y \xrightarrow{ev} Y$$

$$\text{ie- } \begin{array}{ccccc} \Omega Z \times \Sigma A & \xrightarrow{\tilde{\gamma}} & E & \longrightarrow & E' \\ \downarrow \cong & & \downarrow p & & \\ (\Omega Z \times \Sigma A) \vee \Sigma A & \xrightarrow{[\gamma, f] + f} & Y & & \end{array}$$

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