Inert Maps and Connected Sums

Just saw $M \in \mathcal{M}_n$ with htpy cofibrations

$$S^{n-1} \to S^m \vee S^{n-m} \vee J \xrightarrow{j} M$$

$$J \to M \to Q$$

where $H^*(Q) \cong H^*(S^m \times S^{n-m})$

has Ωj has a right htpy inverse.

This is reminiscent to something from rational htpy theory.

Def: Rationalize spaces and maps. A cell attachment

$$S^{n-1} \xrightarrow{f} X \xrightarrow{j} Y = X \cup e^n$$

The map f is <u>inert</u> if $(\Omega j)_*$ is an epimorphism.

Ex: $S^{m+n-1} \xrightarrow{f} S^m \vee S^n \xrightarrow{j} S^m \times S^n$

Whitehead

product

$(\Omega j)_*$ is an epimorphism $\Rightarrow$ f is inert.

Ex: (Halperin-Lemaire) Let M be a closed, oriented manifold. Then the attaching map for the top cell

$$S^{n-1} \xrightarrow{f} M_{n-1} \xrightarrow{j} M$$

is inert.

Observe: $\Omega X \xrightarrow{j} \Omega Y$

Rationally, ΩX, ΩY are htpy equivalent to products of spheres and loops on spheres.

$\Rightarrow$ $(\Omega j)_*$ is an epimorphism

implies Ωj has a right htpy inverse.

This suggests the definition of inert map can be generalised.

Def: (Integral spaces) Consider a htpy cofibration of 1-connected spaces

$$A \xrightarrow{f} X \xrightarrow{j} Y$$

The map f is <u>inert</u> if Ωj has a right htpy inverse.

<u>Ex</u>: M 1-connected n-dimensional PD complex in $\mathcal{M}_n$, $\exists$ htpy cofib

$$S^{n-1} \xrightarrow{f} S^m \vee S^{n-m} \vee J \xrightarrow{j} M$$

where Ωj has a right htpy inverse so f is inert.

We'll use this to study connected sums.

Generally, $M \# N$ is important in geometry but hard to handle in htpy theory.

Nevertheless, it is natural to ask about

$$\pi_*(M \# N).$$

In the manifold case, if M, N are n-dim oriented closed manifolds then $\exists$ htpy cofibrations

$$S^{n-1} \xrightarrow{f} M_{n-1} \to M$$

$$S^{n-1} \xrightarrow{g} N_{n-1} \to N.$$

The connected sum satisfies a htpy cofibration

$$S^{n-1} \xrightarrow{f+g} M_{n-1} \vee N_{n-1} \to M \# N.$$

Generalize.

<u>Def</u>: Suppose $\exists$ htpy cofibrations $\Sigma A \xrightarrow{f} X \to M$, $\Sigma A \xrightarrow{g} Y \to N$. The <u>connected sum over ΣA</u> is given by the htpy cofibration

$$\Sigma A \xrightarrow{f+g} X \vee Y \to M \#_{\Sigma A} N.$$

<u>Thm</u>: Suppose $\exists$ htpy cofibrations $\Sigma A \xrightarrow{f} X \to M$, $\Sigma A \xrightarrow{g} Y \to N$. If f is inert then:

① $\Omega(M \#_{\Sigma A} N) \simeq \Omega M \times \Omega(\Omega M \ltimes Y)$

② $f+g$ is inert

③ $\exists$ htpy fibration

$$\Omega(M\#_{\Sigma A}N) \ltimes S^{n-1} \longrightarrow X \vee Y \longrightarrow M \#_{\Sigma A} N$$

that splits after looping.

<u>Pf</u>: Argument due to Sebastian Chenery.

Consider the htpy pushout

$$\begin{tikzcd}
 & Y \arrow[r, equal] \arrow[d] & Y \arrow[d] \\
\Sigma A \arrow[r, "f+g"] \arrow[d, equal] & X \vee Y \arrow[r] \arrow[d, "q"] & M \#_{\Sigma A} N \arrow[d, "q'"] \\
\Sigma A \arrow[r, "f"] & X \arrow[r, "j"] & M
\end{tikzcd}$$

q = pinch map, q' = induced map.

- q has a right htpy inclusion
- Ωj has a right htpy inverse because f is inert.

$\Rightarrow$

$$\begin{tikzcd}
\Omega M \arrow[r, "s"] \arrow[d, dash] & \Omega X \arrow[r, "\Omega i_1"] \arrow[d, equal] & \Omega(X \vee Y) \arrow[r] \arrow[d, "\Omega q"] & \Omega(M \#_{\Sigma A} N) \arrow[d, "\Omega q'"] \\
 & & &
\end{tikzcd}$$

$$\Omega X \xrightarrow{\Omega j} \Omega M'$$

$=$

So $\Omega q'$ has a right htpy inverse.

By Cor C, $\exists$ htpy fibration

$$\Omega M \rtimes Y \longrightarrow M \#_{\Sigma A} N \xrightarrow{q'} M$$

and a htpy equivalence

$$\Omega(M \#_{\Sigma A} N) \simeq \Omega M \times (\Omega M \rtimes Y).$$

– This proves part ⓐ.

The naturality property gives a htpy fib dgrm

$$\begin{tikzcd}
\Omega X \rtimes Y \arrow[r, "\Omega j \rtimes 1"] \arrow[d] & \Omega M \rtimes Y \arrow[d] \\
X \vee Y \arrow[r, "i"] \arrow[d, "q"] & M \#_{\Sigma A} N \arrow[d, "q'"] \\
X \arrow[r, "j"] & M
\end{tikzcd}$$

$\Rightarrow \Omega i$ has a right htpy inverse.

$\Rightarrow$ $f+g$ is inert.

Then Cor C also gives a htpy fib

$$\Omega(M \#_{\Sigma A} N) \ltimes S^{n-1} \to X \vee Y \xrightarrow{i} M \#_{\Sigma A} N$$

that splits after looping. $\square$

Special case: connected sums of n-dim closed, oriented manifolds.

Cofibrations $\quad S^{n-1} \xrightarrow{f} M_{n-1} \to M$

$\quad S^{n-1} \to N_{n-1} \to N$

If f is inert then

$$\Omega(M \# N) \simeq \Omega M \times \Omega(\Omega M \ltimes N_{n-1})$$

<u>Ex</u>: Inertness condition holds if M is an $(n-1)$-connected $2n$-dim mnfld, $n \notin \{4, 8\}$, and $H^n(M) \cong \mathbb{Z}^d$ for $d \geq 2$.

Then let N be any closed, oriented n-dim mnfld (1-connected)

Then get decomp for $\Omega(M \# N)$

$\Rightarrow$ info on $\pi_*(M \# N)$.