Recall an n-dimensional PD complex is in class $\mathcal{M}_n$ if there exists htpy cofibrations

$$S^{n-1} \to S^m \vee S^{n-m} \vee J \to M$$

$$J \to M \to Q$$

where $H^*(Q) \cong H^*(S^m \times S^{n-m})$

We showed $\exists$ htpy fibration

$$\Omega Q \ltimes J \to M \to Q$$

and $\Omega Q \simeq \Omega S^m \times \Omega S^{n-m}$, and

$$\Omega M \simeq \Omega Q \times \Omega(\Omega Q \ltimes J).$$

Is there something to say about the cofib

$$S^{n-1} \to S^m \vee S^{n-m} \vee J \to M \ ?$$

<u>A Naturality Property</u>

Cor C stated if $\exists$ htpy cofibration $A \xrightarrow{f} B \xrightarrow{h} D$

of 1-connected spaces and if Ωh has a right htpy inverse then $\exists$ htpy fibration

$$\circledast \quad \Omega D \ltimes A \longrightarrow B \xrightarrow{h} D$$

and a htpy equivalence

$$\circledast\circledast \quad \Omega B \simeq \Omega D \times \Omega(\Omega D \ltimes A).$$

<u>Thm</u>: The fibration $\circledast$ is natural for maps of htpy cofibrations

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[r, "h"] \arrow[d] & D \arrow[d] \\ A' \arrow[r, "f'"] & B' \arrow[r, "h'"] & D' \end{tikzcd}$$

with compatible lifts

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d] & \mathrm{Fibre}(h) \arrow[d] \\ A' \arrow[r, "g'"] & \mathrm{Fibre}(h') \end{tikzcd}$$

and compatible sections

$$\Omega D \longrightarrow \Omega B$$

$$\begin{array}{ccc} \downarrow & & \downarrow \\ \Omega D' & \longrightarrow & \Omega B' . \end{array}$$

Pf: Define F, F' be defined by the htpy fibration dgrm

$$\begin{array}{ccccc} F & \longrightarrow & B & \xrightarrow{h} & D \\ \downarrow & & \downarrow & & \downarrow \\ F' & \longrightarrow & B' & \xrightarrow{h'} & D' . \end{array}$$

The identification $F \simeq \Omega D \ltimes A$ was obtained from a composite

$$\Omega D \times A \xrightarrow{1 \times g} \Omega D \times F \xrightarrow{act} F$$

and a quotient from sth having a right htpy inverse:

$$\Omega D \ltimes A \longrightarrow F .$$

This is natural if $\exists$ map of fibration diagrams (for the action) and compatible lifts g, g'. Also need compatible quotients - comes from the compatible sections.

The htpy equivalence for ΩB came from the composite

$$\Omega D \times \Omega(\Omega D \ltimes A) \xrightarrow{s \times \Omega j} \Omega B \times \Omega B \xrightarrow{\mu} \Omega B$$

s is a right htpy inverse for $\Omega B \xrightarrow{\Omega h} \Omega D$

$j : \Omega D \ltimes A \xrightarrow{\simeq} F \to B.$

This is natural for compatible sections s, s' and the compatibility of $\Omega j, \Omega j'$ from the naturality of the decomposition of F.

$\square$

<u>Applying the Naturality Property</u>

$M \in \mathcal{M}_n$

$\exists$ htpy cofibrations

$$S^{n-1} \to S^m \vee S^{n-m} \vee J \to M$$

$$J \to M \to Q$$

with

$$H^*(Q) \cong H^*(S^m \times S^{n-m}).$$

Consider the htpy cofibration diagram

$$\begin{tikzcd}
J \arrow[r, "f"] \arrow[d, equal] & S^m \vee S^{n-m} \vee J \arrow[r, "h"] \arrow[d] & S^m \vee S^{n-m} \arrow[d] \\
J \arrow[r, "f'"] & M \arrow[r, "h'"] & Q
\end{tikzcd}$$

- $\exists$ lift

$$\begin{tikzcd}
 & \mathrm{Fibre}(h) \arrow[d] \\
J \arrow[ur, dashed, "g"] \arrow[r, "f"] & S^m \vee S^{n-m} \vee J
\end{tikzcd}$$

for appropriate g.

- Define $g' : J \xrightarrow{g} \mathrm{Fibre}(h) \longrightarrow \mathrm{Fibre}(h')$

- Get

$$\begin{tikzcd}
J \arrow[r, "g"] \arrow[d, equal] & \mathrm{Fibre}(h) \arrow[d] \\
J \arrow[r, "g'"] & \mathrm{Fibre}(h')
\end{tikzcd}$$

- Compatible sections also follow from ΩQ retracting off $\Omega(S^m \vee S^{n-m})$.

The Naturality Property gives a htpy fibration diagram

$$\begin{tikzcd}
\Omega(S^m \vee S^{n-m}) \ltimes J \arrow[r] \arrow[d, "\Omega i \ltimes 1"] & S^m \vee S^n \vee J \arrow[r] \arrow[d, "j"] & S^n \vee S^{n-m} \arrow[d, "i"] \\
 & &
\end{tikzcd}$$

$$\Omega Q \rtimes J \longrightarrow M \xrightarrow{h'} Q$$

Thm: Ωj has a right htpy inverse.

Pf:

$$\begin{tikzcd}
\Omega(\Omega Q \rtimes J) \arrow[d, "\Omega(s \rtimes 1)"] \arrow[dd, bend right, "="'] & & \Omega Q \arrow[d, "s"] \arrow[dd, bend left, "="] \\
\Omega(\Omega(S^m \vee S^{n-m}) \rtimes J) \arrow[r] \arrow[d, "\Omega(\Omega i \rtimes 1)"] & \Omega(S^m \vee S^{n-m} \vee J) \arrow[r] \arrow[d, "\Omega j"] & \Omega(S^m \vee S^{n-m}) \arrow[d, "\Omega i"] \\
\Omega(\Omega Q \rtimes J) \arrow[r] & \Omega M \arrow[r] & \Omega Q
\end{tikzcd}$$

Have $\Omega M \simeq \Omega Q \times \Omega(\Omega Q \rtimes J)$

Get $\Omega(\Omega Q \rtimes J), \Omega Q \rightsquigarrow \Omega M$

factor "disjointly" through Ωj.

$\Rightarrow$ ΩM retracts off $\Omega(S^m \vee S^{n-m} \vee J)$

ie. Ωj has a right htpy inverse.

$\square$

Ex: M is an $(n-1)$-connected $2n$-dim PD complex, $n \notin \{2,4,8\}$, and $H^n(M) \cong \mathbb{Z}^d$ for $d \geq 2$. Then

$$S^{2n-1} \longrightarrow \bigvee_{i=1}^{d} S^n \xrightarrow{j} M$$

As $M \in \mathcal{M}_{2n}$, Ωj has a right htpy inverse

$$\Rightarrow \Omega\left(\bigvee_{i=1}^{d} S^n\right) \simeq \Omega M \times \Omega(\Omega M \ltimes S^{2n-1}).$$

(Note: $\exists$ fibration

$$\Omega M \ltimes S^{2n-1} \longrightarrow \bigvee_{i=1}^{n} S^n \xrightarrow{j} M \quad \text{by Cor C)}$$

Ex: M is an $(n-1)$-connected $(2n+1)$-dim PD complex with $H^n(M) = \mathbb{Z}^d$ for $d \geq 1$.

$\exists$ htpy cofibration

$$S^{2n} \longrightarrow \bigvee_{i=1}^{d} (S^n \vee S^{n+1}) \vee M(T, n) \xrightarrow{j} M.$$

$M \in \mathcal{M}_{2n+1}$

$\Rightarrow \Omega j$ has a right htpy inverse

$\Rightarrow$ By Cor C, $\exists$ htpy fibration

$$\Omega M \ltimes S^{2n} \longrightarrow \bigvee^{d} (S^n \vee S^{n+1}) \vee M(T, n) \xrightarrow{j} M$$

$i=1$

and $\Omega\left(\bigvee_{i=1}^{d}(S^n \vee S^{n+1}) \vee M(7,n)\right) \simeq \Omega M \times \Omega(\Omega M \ltimes S^{2n})$.

<u>Application</u>:

- Modified version works for 1-connected 4-PD-complexes.
  ie - htpy cofibration

$$S^3 \longrightarrow \bigvee_{i=1}^{d} S^2 \xrightarrow{j} M$$

- Ωj has a right htpy inverse

G = 1-connected, simple, compact Lie gp.

$[M, G] \cong \mathbb{Z} \Rightarrow \exists$ countably many distinct principle G-bundles over M

$$P_k \longrightarrow M$$

classified by $M \xrightarrow{\alpha_k} BG$.

Observe
$$\begin{tikzcd} \bigvee_{i=1}^{d} S^2 \arrow[r, hook, "j"] \arrow[rr, bend right] & M \arrow[r, "\alpha_k"] & BG \end{tikzcd}$$

$\simeq *$ as BG is 2-connected.

But Ωj has a right inverse $s: \Omega M \to \Omega(\bigvee_{i=1}^{d} S^3)$

so $\Omega\alpha_k \simeq \Omega\alpha_k \circ \Omega j \circ s \simeq * \circ s \simeq *$.

$\Rightarrow$ fibration seq

$$\Omega P_k \to \Omega M \xrightarrow{\Omega\alpha_k} G \to P_k \to M \xrightarrow{\alpha_k} BG$$

gives $\Omega P_k \simeq \Omega M \times \Omega G \quad \forall k$.

$\Rightarrow \quad \pi_*(P_k) \cong \pi_*(P_0) \quad \forall k$.