Example: Poincaré Duality Complexes

Def: A finite CW-complex M is a Poincaré Duality complex if $H^*(M;R)$ satisfies Poincaré Duality for any coefficient ring R.

Ex: Any closed, oriented manifold is a PD-complex.

Let $\mathcal{M}_n$ be the class of n-dimensional Poincaré Duality complexes such that:

- $\exists$ htpy cofibration

$$S^{n-1} \xrightarrow{f} \underset{\substack{\cong \\ M_{n-1}}}{S^m \vee S^{n-m} \vee J} \longrightarrow M$$

where f attaches the top cell to M, J = some space.

- $\exists$ htpy cofibration

$$J \hookrightarrow M \longrightarrow Q$$

where $H^*(Q) \cong H^*(S^m \times S^{n-m})$

<u>Ex</u>: Let M be an $(n-1)$-connected $2n$-dimensional PD complex. Then by Poincaré Duality

$$H^m(M) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0, 2n \\ \mathbb{Z}^d & \text{if } m = n, \text{ some } d \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

$\exists$ htpy cofibration

$$S^{2n-1} \xrightarrow{f} \bigvee^d S^n \longrightarrow M.$$

- if $d = 0$ then $M \simeq S^{2n}$.
- if $d = 1$ then $n \in \{2, 4, 8\}$ - Hopf Inv 1
- Assume $d \geq 2$.

  Assume $n \in \{2, 4, 8\}$.

  Then $\exists$ 2 distinct copies of S^n in $\bigvee^d S^n$ s.t. the corresponding generators in cohlgy have cup product equal to the Fundamental class

$\Rightarrow$ $\exists$ htpy cofibrations

$$S^{2n-1} \xrightarrow{f} S^n \vee S^n \vee J \longrightarrow M \qquad \left(J = \bigvee^{d-2} S^n\right)$$

and $J \hookrightarrow M \longrightarrow Q$

where $H^*(Q) = H^*(S^n \times S^n)$.

$\Rightarrow$ $M \in \mathcal{M}_{2n}$.

<u>Ex</u>: Let M be an $(n-1)$-connected $(2n+1)$-dim PD complex. Then

$$H^m(M) \cong \begin{cases} \mathbb{Z} & \text{if } m = 0, 2n+1 \\ \mathbb{Z}^d & \text{if } m = n \\ \mathbb{Z}^d \oplus T & \text{if } m = n+1 \\ 0, & \text{otherwise} \end{cases}$$

where T = torsion.

$\exists$ htpy cofibration

$$S^{2n} \xrightarrow{f} \left(\bigvee^{d} (S^n \vee S^{n+1}) \right) \vee M(T, n) \longrightarrow M.$$

$M(T,n)$ = Moore space with $H^{n+1}(M(T,n)) = T$.

If $d \geq 1$ then $\exists$ pair $S^n \vee S^{n+1}$ such that

$\exists$ htpy cofibrations

$$S^{2n} \xrightarrow{f} S^n \vee S^{n+1} \vee J \longrightarrow M \qquad \left(J = \left(\bigvee^{d-1} S^n \vee S^{n+1}\right) \vee M(T,n)\right)$$

and

$$J \hookrightarrow M \longrightarrow Q$$

where $H^*(Q) = H^*(S^n \times S^{n+1})$.

$$\Rightarrow M \in \mathcal{M}_{2n+1}$$

<u>Loop space Decomposition for $M \in \mathcal{M}_n$.</u>

Given $\exists$ htpy cofibrations

$$S^{n-1} \xrightarrow{f} S^m \vee S^{n-m} \vee J \longrightarrow M$$

$$J \hookrightarrow M \xrightarrow{h} Q$$

where $H^*(Q) \cong H^*(S^m \times S^{n-m})$.

Dually, $H_*(Q) = H_*(S^m \times S^{n-m})$

Observe – A Serre spectral sequence calculation shows

$$H_*(\Omega Q) \cong H_*(\Omega S^m \times \Omega S^{n-m})$$

Consider

$$\begin{tikzcd}
\Omega S^m \times \Omega S^{n-m} \arrow[d] \arrow[ddrr, "\lambda"] & & \\
\Omega(S^m \vee S^{n-m}) \arrow[d] & & \\
\Omega(S^m \vee S^{n-m} \vee J) \arrow[r] & \Omega M \arrow[r, "\Omega h"] & \Omega Q
\end{tikzcd}$$

λ_* is a homology isomorphism.

$\Rightarrow$ λ is a htpy equivalence

$\Rightarrow$ Ωh has a right htpy inverse.

Apply Cor C to get:

<u>Thm</u>: Let $M \in \mathcal{M}_n$. Then $\exists$ htpy fibration

$$\overset{\Omega Q \ltimes J}{\cancel{\Omega Q \ltimes S^{n-1}}} \to M \xrightarrow{h} Q$$

and $\Omega M \simeq \Omega Q \times \Omega(\Omega Q \ltimes \overset{J}{\cancel{S^{n-1}}})$.

Also $\Omega Q \simeq \Omega S^m \times \Omega S^{n-m}$.

Note: $\Omega Q \ltimes S^{n-1} \simeq (\Omega Q \wedge S^{n-1}) \vee S^{n-1}$

$$\simeq \left((\Omega S^m \times \Omega S^{n-m}) \wedge S^{n-1} \right) \vee S^{n-1}$$

$\simeq$ wedge of spheres, W.

Rewrite: $\Omega M \simeq \Omega S^m \times \Omega S^{n-m} \times \Omega W$

Ex: If M is an $(n-1)$-connected $2n$-dim PD complex, $n \in \{2,4,8\}$, $d \geq 2$ for $H^n(M) = \mathbb{Z}^d$. Then $\exists$ htpy fibration

$$\Omega Q \ltimes J \longrightarrow M \longrightarrow Q \qquad \left(J = \bigvee^{d-2} S^n \right)$$

$$\Omega M \simeq \Omega Q \times \Omega(\Omega Q \ltimes J)$$

and $\Omega M \simeq \Omega S^m \times \Omega S^{n-m} \times \Omega W$

wedge of spheres.

Ex: If M is an $(n-1)$-connected $(2n+1)$-dim PD complex, $d \geq 1$ for $H^n(M) \cong \mathbb{Z}^d$, then $\exists$ htpy fibration

$$\Omega Q \ltimes J \longrightarrow M \longrightarrow Q$$

$$J = \left(\bigvee^{d-1} S^n \vee S^{n+1} \right) \vee M(T,n)$$

and $\Omega M \simeq \Omega S^m \times \Omega S^{n+1} \times \Omega\left((\Omega S^n \times \Omega S^{n+1}) \ltimes J\right)$.

$\simeq \Omega S^n \times \Omega S^{n+1} \times \Omega W'$

where W' = wedge of spheres and Moore spaces

Consequences.

Moore's Conjecture holds for $S^a \vee S^b$.
So it holds for any M where
$\Omega(S^a \vee S^b)$ retracts off ΩM.

Also shows M is mod-p^r hyperbolic $\forall p, \forall r \geq 1$.

Ex: M is an $(n-1)$-connected $2n$-dim PD complex,
$n \notin \{2,4,8\}$, $H^n(M) \cong \mathbb{Z}^d$ for $d \geq 2$.
Then

$$\Omega M \simeq \Omega S^n \times \Omega S^n \times \Omega W$$

where W is a wedge of spheres.

$d = 2$ $\quad \Omega M \simeq \Omega S^m \times \Omega S^n$ - elliptic, has exp at all p.

$d \geq 3$ W has at least 2 spheres in it.

$\Rightarrow$ M is hyperbolic, no exp at any prime p, mod-p^r hyperbolic $\forall p, \forall r \geq 1$.

<u>Ex</u>: M is an $(n-1)$-connected $(2n+1)$-dim PD complex, $H^n(M) \cong \mathbb{Z}^d$, $d \geq 1$ then

$$\Omega M \simeq \Omega S^n \times \Omega S^{n+1} \times \Omega W'$$

where W' = wedge of spheres and Moore spaces.

If $d = 1$ then W' = wedge of Moore spaces.

(Assume no mod 2-Moore spaces)

$\Rightarrow$ Moore's Conj, mod-p^r hyperbolicity certain r

If $d > 1$ then $\exists$ 2 spheres retracting off W' $\Rightarrow$ hyperbolic, no exponent at any prime p, mod-p^r hyperbolic $\forall p, \forall r \geq 1$.

Rigidity: $(n-1)$-connected, $2n$-dim PD complex.
$H^n(M) \cong \mathbb{Z}^d$, $d \geq 2$.

$$\Omega M \simeq \Omega S^n \times \Omega S^n \times \Omega\Big((\Omega S^n \times \Omega S^n) \ltimes \big(\bigvee^{d-2} S^n\big)\Big)$$

$\Rightarrow$ htpy type of ΩM depends only on d.

Cor: $\Omega M \simeq \Omega M'$ iff $H^n(M) \cong H^n(M')$.

$\Rightarrow \pi_*(M) \cong \pi_*(M')$ iff $H^n(M) \cong H^n(M')$.

Extra Note: 1-connected 4-dim PD complex M.

Duan-Liang: $\Omega M \simeq S^1 \times \Omega\big((S^2 \times S^3)^{\# d-1}\big)$.

where $H^2(M) \cong \mathbb{Z}^d$.

Can decompose $(S^2 \times S^3)^{\# d-1}$ - a 1-connected 5-dim PD complex

$$\Rightarrow \Omega M \simeq S^1 \times \Omega S^2 \times \Omega S^3 \times \Omega\Big((\Omega S^2 \times \Omega S^3) \ltimes \big(\bigvee^{d-2} S^2 \vee S^3\big)\Big)$$