A Decomposition Method

Return to the Cube Lemma:

$$\begin{tikzcd}
E \arrow[rr, "a"] \arrow[dr, "b"] \arrow[dd] & & F \arrow[dr] \arrow[dd] & \\
& G \arrow[rr, crossing over] & & H \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& C \arrow[rr] \arrow[from=uu, crossing over] & & D \arrow[d] \\
& & & Z
\end{tikzcd}$$

- bottom face is a htpy pushout
- 4 sides are htpy pullbacks

$\Rightarrow$ top face is a htpy pushout.

Tricky part: finding info on a, b to find the htpy type of H.

Specialize: Suppose $\exists$ $D \to Z$ inducing the cube -ie- $\exists$ htpy fibration

$$H \to D \to Z$$

and

$$F \to B \to Z$$

$$G \longrightarrow C \longrightarrow Z$$

$$E \longrightarrow A \longrightarrow D.$$

Advantage: $\exists$ compatible htpy action of ΩZ on H, F, G, E

Specialize more:

<u>Thm A</u> (Gray-T, Beben-T)

Suppose $\exists$ htpy cofibration $A \xrightarrow{f} B \longrightarrow D$ and a map $D \to Z$. Then $\exists$ htpy commutative cube

$$\begin{tikzcd}
\Omega Z \times A \arrow[rr, "f"] \arrow[dr, "\pi"] \arrow[dd] & & F \arrow[dr] \arrow[dd] & \\
& \Omega Z \arrow[rr] \arrow[dd] & & H \arrow[dd] \\
A \arrow[rr, "f"] \arrow[dr] & & B \arrow[dr] & \\
& * \arrow[rr] & & D \arrow[d] \\
& & & Z
\end{tikzcd}$$

where
- 4 sides are htpy pullbacks
- top face is a htpy pushout
- π_1 is a projection

- for an appropriate lift

$$\begin{tikzcd} & F \arrow[d] \\ A \arrow[ur, dashed, "g"] \arrow[r, "f"] \arrow[dr, bend right, "*"'] & B \arrow[d] \\ & Z \end{tikzcd}$$

φ is the composite

$$\Omega Z \times A \xrightarrow{1 \times g} \Omega Z \times F \xrightarrow{act} F.$$

<u>Remarks</u>: ① Observe

$$\begin{tikzcd} A \arrow[d, hook] \arrow[dr, bend left, "g"] & \\ \Omega Z \times A \arrow[r, "\varphi"] \arrow[d, "\pi_1"] & F \arrow[d] \\ \Omega Z \arrow[r] & H \end{tikzcd}$$

$\Rightarrow$ Need lift g such that $A \xrightarrow{g} F \to H$ is null htpic.

$\Rightarrow$ g lifts to the fibre of $F \to H$
$$\shortparallel$$
fibre of $B \to D$

Note $\exists$ lift

$$\begin{tikzcd} & X \arrow[d] \\ A \arrow[ur, dashed, "f'"] \arrow[r, "f"] & B \end{tikzcd}$$

$$\begin{tikzcd} * \arrow[dr] & \arrow[d] \\ & D \end{tikzcd}$$

Choose g to be $A \xrightarrow{f'} X \longrightarrow F$

This is an appropriate choice of g, always exists.

(2) Sketch of proof:
Replace htpy cofib $A \xrightarrow{f} B \to D$ by a real cofibration

$$A \xrightarrow{f} B \longrightarrow B \cup_f CA$$

Replace fibration $H \to D \to Z$ by

$$\begin{tikzcd} Q \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{pullback}"] & PZ \arrow[d, "ev"] \\ B \cup_f CA \arrow[r] & Z \end{tikzcd}$$

PZ = path space

ie - fibration $Q \to B \cup_f CA \to Z$

Carefully analyze the maps in the cube.

Going further: Suppose $B \xrightarrow{h} Z$ (ie - $B \to D \to Z$)

has the property that Ωh has a right htpy inverse - ie - ΩZ retracts off ΩB.

Then in the htpy fibration sequence

$$\Omega Z \xrightarrow{\partial} F \longrightarrow B \xrightarrow{h} Z$$

∂ is null homotopic so

$$\begin{tikzcd} \Omega Z \times F \arrow[r, "act"] \arrow[d] & F \\ \Omega Z \ltimes F \arrow[ur, dashed] & \end{tikzcd}$$

$\nwarrow$ half-smash

$= (\Omega Z \times F) / (\Omega Z \times \{*\})$.

$\Rightarrow \Omega Z \to H$ is also null htpic

$\Rightarrow$ pushout

$$\begin{tikzcd} \Omega Z \ltimes A \arrow[r, "\varphi"] \arrow[d, "\pi_1"] & F \arrow[d] \\ \Omega Z \arrow[r] & H \end{tikzcd}$$

can have ΩZ pinched out to a new htpy pushout

$$\Omega Z \ltimes A \xrightarrow{\bar{\varphi}} F$$

$$\begin{tikzcd} {} \arrow[d] & {} \arrow[d] \\ * \arrow[r] & H \end{tikzcd}.$$

-ie- $\exists$ htpy cofibration

$$\Omega Z \ltimes A \xrightarrow{\bar{\ell}} F \longrightarrow H.$$

<u>Thm B</u>: Suppose $\exists$ htpy cofibration $A \xrightarrow{f} B \longrightarrow D$ and a map $D \longrightarrow Z$ such that $h: B \longrightarrow D \longrightarrow Z$ has Ωh with a right htpy inverse.
Then $\exists$ htpy cofibration

$$\Omega Z \ltimes A \xrightarrow{\bar{\ell}} F \longrightarrow H.$$

Specialise more:

<u>Cor C</u>: Suppose $\exists$ htpy cofibration $A \xrightarrow{f} B \xrightarrow{h} D$ where Ωh has a right htpy inverse.
Then $\exists$ htpy fibration

$$\Omega Z \ltimes A \longrightarrow B \xrightarrow{h} D$$

and $\exists$ htpy equivalence

$$\Omega B \simeq \Omega D \times \Omega(\Omega Z \ltimes A).$$

Note: Assume throughout spaces are 1-connected.

Pf: By Thm A, $\exists$ htpy commutative cube

$$\begin{tikzcd}
\Omega D \times A \arrow[rr, "\varphi"] \arrow[dr, "\pi_1"] \arrow[dd] & & F \arrow[dr] \arrow[dd] & \\
& \Omega D \arrow[rr] \arrow[dd] & & * \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& * \arrow[rr] & & D \arrow[d, "="] \\
& & & D
\end{tikzcd}$$

where the top face is a htpy pushout. Since Ωh has a right htpy inverse, Thm B says $\exists$ htpy cofibration

$$\Omega D \ltimes A \xrightarrow{\bar{\varphi}} F \longrightarrow * .$$

$\Rightarrow$ $\bar{\varphi}_*$ is iso

$\Rightarrow$ $\bar{\varphi}$ is a htpy equivalence by Whitehead's Thm.

ie. the fibration $F \to B \xrightarrow{h} D$

is $\qquad \Omega D \ltimes A \longrightarrow B \xrightarrow{h} D.$

Also, Ωh has a right htpy inverse

$\Rightarrow \Omega B \simeq \Omega D \times \Omega(\Omega D \ltimes A).$ $\square$

Cor C is very powerful.

<u>Ex</u>: (Known) $\exists$ htpy cofibration

$$X \hookrightarrow X \vee Y \xrightarrow{q} Y$$

Then q has a right inverse, so Ωq does too

$\Rightarrow$ Cor C says $\exists$ htpy fibration

$$\Omega Y \ltimes X \longrightarrow X \vee Y \xrightarrow{q} Y.$$

and $\quad \Omega(X \vee Y) \simeq \Omega Y \times \Omega(\Omega Y \ltimes X).$

<u>Refinements</u>: ① In general, $\exists$ htpy equivalence

$$A \ltimes \Sigma B \simeq (A \wedge \Sigma B) \vee \Sigma B$$

So a htpy cofibration $\Sigma A \xrightarrow{f} B \xrightarrow{h} D$ where Ωh has a right htpy inverse then $\exists$ htpy fibration

$$\Omega D \ltimes \Sigma A \longrightarrow B \xrightarrow{h} D$$

$$\Omega D \ltimes \Sigma A \cong (\Omega D \wedge \Sigma A) \vee \Sigma A$$

② If $D = \Sigma \bar{D}$ then $\exists$ htpy fibration

$$\Omega \Sigma \bar{D} \ltimes \Sigma A \longrightarrow B \xrightarrow{h} \Sigma \bar{D}$$

$$\Omega \Sigma \bar{D} \ltimes \Sigma A \cong (\Omega \Sigma \bar{D} \wedge \Sigma A) \vee \Sigma A$$

$$\Omega \Sigma \bar{D} \wedge \Sigma A \simeq (\Sigma \Omega \Sigma \bar{D}) \wedge A$$

$$\simeq \Big(\bigvee_{k \geq 1} \Sigma \bar{D}^{\wedge k} \Big) \wedge A$$