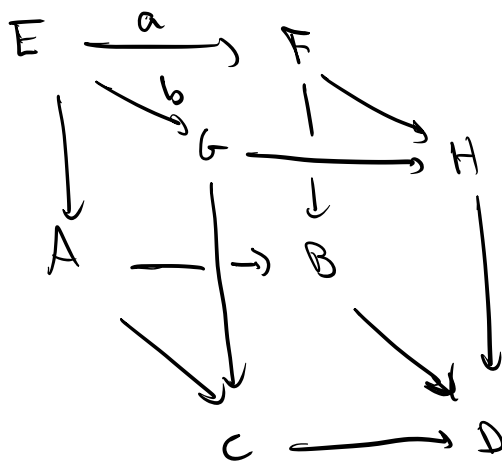


A Decomposition Method

Return to the Cube Lemma:



- bottom face is a htpy product
- 4 sides are htpy pullbacks
- \Rightarrow top face is a htpy product.

Tricky part: finding info on a, b to find the htpy type of H .

Specialize: Suppose $\exists D \rightarrow Z$ inducing the cube -ie- $\exists \text{htpy}$ fibration

$$H \rightarrow D \rightarrow Z$$

and $F \rightarrow B \rightarrow Z$

$$\begin{array}{ccc} G & \rightarrow & C \rightarrow Z \\ E & \rightarrow & A \rightarrow D. \end{array}$$

Advantage: \exists htpy action of ΩZ on H, F, G, E
 \wedge
 compatible.

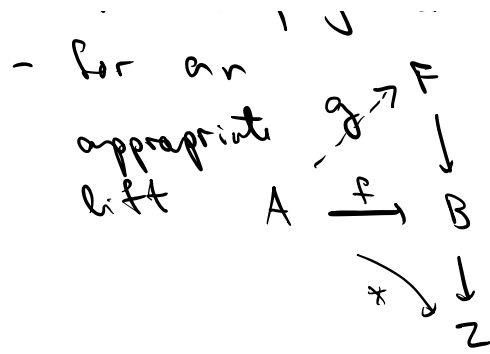
Specialize more:

Thm A (Gray-T, Beben-T)

Suppose \exists htpy cofibration $A \xrightarrow{f} B \rightarrow D$
 and a map $D \rightarrow Z$. Then \exists htpy commutative
 cube

$$\begin{array}{ccccc} \Omega Z \times A & \xrightarrow{f} & F & & \\ \downarrow & \searrow \pi & \downarrow & \searrow & \\ \Omega Z & & \Omega Z & \xrightarrow{\quad} & H \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & B & & D \\ & \searrow & \searrow & \searrow & \downarrow \\ & & * & \xrightarrow{\quad} & D \\ & & & & \downarrow \\ & & & & Z \end{array}$$

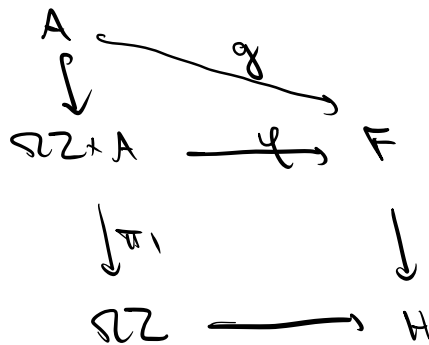
- where - 4 sides are htpy pullbacks
 - top face is a htpy pushout
 - π is a projection



φ is the composite

$$\Omega Z \times A \xrightarrow{1 \times g} \Omega Z \times F \xrightarrow{act} F.$$

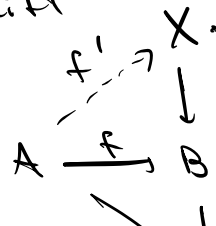
Remarks: ① Observe



\Rightarrow Need lift g such that $A \xrightarrow{g} F \rightarrow H$ is null homotopic.

$\Rightarrow g$ lifts to the fibre of $F \rightarrow H$
 " fibre of $B \rightarrow D$

Note \exists lift



$$x \downarrow \downarrow \\ D$$

Choose g to be $A \xrightarrow{f'} X \rightarrow \mathbb{R}$

This is an appropriate choice of g ,
always exists.

② Sketch of proof:

Replace htpy cofib $A \xrightarrow{f} B \rightarrow D$
by a real cofibration

$$A \xrightarrow{f} B \rightarrow BU_f CA$$

Replace fibration $H \rightarrow D \rightarrow Z$ by

$$\begin{array}{ccc} Q & \longrightarrow & PZ = \text{path space} \\ \downarrow \text{pullback} & & \downarrow \text{ev} \\ BU_f CA & \longrightarrow & Z \end{array}$$

-ie - fibration $Q \rightarrow BU_f CA \rightarrow Z$

Carefully analyze the maps in the cube.

Going further: Suppose $B \xrightarrow{h} Z$ (ie - $B \rightarrow D \rightarrow Z$)

has the property that ΩZ has a right homotopy inverse - ie - ΩZ retracts off ΩB .

Then in the homotopy fibration sequence

$$\Omega Z \xrightarrow{d} F \rightarrow B \xrightarrow{h} Z$$

d is null homotopic so

$$\begin{array}{ccc} \Omega Z \times F & \xrightarrow{\text{act}} & F \\ \downarrow & \nearrow & \\ \Omega Z \rtimes F & & \end{array}$$

\nwarrow half-smash
 $= (\Omega Z \times F) / (\Omega Z \times \{*\})$

$\Rightarrow \Omega Z \rightarrow H$ is also null homotopic

\Rightarrow pushout

$$\begin{array}{ccc} \Omega Z \times A & \xrightarrow{\varphi} & F \\ \downarrow \pi_1 & & \downarrow \\ \Omega Z & \longrightarrow & H \end{array}$$

can have ΩZ pinched out to a new homotopy pushout

$$\Omega Z \rtimes A \xrightarrow{\bar{\varphi}} F$$

$$\begin{array}{ccc} & & \\ & \downarrow & \downarrow \\ & * & H \end{array}$$

-ie - \exists htpy cofibration

$$\Omega Z \times A \xrightarrow{\bar{e}} F \rightarrow H.$$

Thm B: Suppose \exists htpy cofibration $A \xrightarrow{f} B \rightarrow D$
and a map $D \rightarrow Z$ such that $h: B \rightarrow D \rightarrow Z$
has Ωh with a right htpy inverse.
Then \exists htpy cofibration

$$\Omega Z \times A \xrightarrow{\bar{f}} F \rightarrow H.$$

Specialise more:

Cor C: Suppose \exists htpy cofibration $A \xrightarrow{f} B \xrightarrow{h} D$
where Ωh has a right htpy inverse.
Then \exists htpy fibration

$$\Omega Z \times A \rightarrow B \xrightarrow{h} D$$

and \exists htpy equivalence

$$\Omega B \simeq \Omega D \times \Omega(\Omega Z \times A).$$

Note: Assume throughout spaces are 1-connected.

Pf: By Thm A, \exists htpy commutative cube

$$\begin{array}{ccccc}
 \Omega D \times A & \xrightarrow{q} & F & & \\
 \downarrow & \searrow \pi_1 & \downarrow & \searrow & \\
 & \Omega D & \downarrow & & * \\
 & & \downarrow & & \\
 A & \xrightarrow{\quad} & B & & \\
 \downarrow & \searrow & \downarrow & \searrow & \\
 & & * & \xrightarrow{\quad} & D \\
 & & & & \downarrow = \\
 & & & & D
 \end{array}$$

where the top face is a htpy pushout.
 Since Ωh has a right htpy inverse,
 Thm B says \exists htpy cofibration

$$\Omega D \times A \xrightarrow{\bar{q}} F \rightarrow *$$

$\Rightarrow \bar{q}_*$ is iso

$\Rightarrow q$ is a htpy equivalence by
 Whitehead's Thm.

ie - the fibration $F \rightarrow B \xrightarrow{h} D$

$$\text{is } \Omega D \times A \rightarrow B \xrightarrow{h} D.$$

Also, Ωh has a right httpy inverse

$$\Rightarrow \Omega B \simeq \Omega D \times \Omega(\Omega D \times A). \quad \square$$

Cor C is very powerful.

Ex: (known) \exists httpy cofibration

$$X \hookrightarrow XY \xrightarrow{g} Y$$

Then g has a right inverse, so Ωg does too

\Rightarrow Cor C says \exists httpy fibration

$$\Omega Y \times X \rightarrow XY \xrightarrow{g} Y.$$

$$\text{and } \Omega(XY) \simeq \Omega Y \times \Omega(\Omega Y \times X).$$

Refinements: ① In general, \exists httpy equivalence

$$A \times \Sigma B \simeq (A \wedge \Sigma B) \vee \Sigma B$$

So a htpy cofibration $\Sigma A \xrightarrow{f} B \xrightarrow{h} D$
 where Sh has a right htpy inverse
 then \exists htpy fibration

$$\begin{array}{ccc} \Omega D \wedge \Sigma A & \longrightarrow & B \xrightarrow{h} D \\ \downarrow \cong & & \\ (\Omega D \wedge \Sigma A) \vee \Sigma A & & \end{array}$$

② If $D = \Sigma \bar{D}$ then \exists htpy fibration

$$\begin{array}{ccc} \Omega \Sigma D \wedge \Sigma A & \longrightarrow & B \xrightarrow{h} \Sigma \bar{D} \\ \downarrow \cong & & \\ (\Omega \Sigma D \wedge \Sigma A) \vee \Sigma A & & \end{array}$$

$$\begin{aligned} \Omega \Sigma D \wedge \Sigma A &\cong (\Sigma \Omega \Sigma D) \wedge A \\ &\cong \left(\bigvee_{k \geq 1} \Sigma D^{\wedge k} \right) \wedge A \end{aligned}$$