Loop Space Decompositions

$\Omega X = \mathrm{Map}_*(S^1, X)$ - based loop space

$\pi_m(\Omega X) = \pi_{m+1}(X)$

Advantage of ΩX: it has a multiplication, is an H-group.

Aim: Decompose ΩX in analogy to decomposing a group.

Traditional Strategy:

① Calculate $H_*(\Omega X)$

② Decompose $H_*(\Omega X)$ as a tensor product

$$H_*(\Omega X) \cong \bigotimes_\alpha M_\alpha$$

iso as coalgebras, modules.

③ Identify M_α as $H_*(X_\alpha)$ for some space X_α.

④ Find a map $X_\alpha \to \Omega X$ inducing

$$M_\alpha \hookrightarrow \bigotimes_\alpha M_\alpha \cong H_*(\Omega X).$$

⑤ Multiply together to get

$$\circledast \quad \prod_\alpha X_\alpha \to \Omega X$$

induces $\bigotimes_\alpha M_\alpha \longrightarrow H_*(\Omega X)$ - an iso -

so $\circledast$ is a htpy equivalence by Whitehead's Theorem.

Steps 1-4 can be hard, Step 5 is easy.

Example: Hilton's Thm:

Decompose $\Omega(S^{m+1} \vee S^{n+1})$ for $m,n \geq 1$.

① Bott-Samelson $\Rightarrow$ (field coeffs)

$$H_*(\Omega(S^{m+1} \vee S^{n+1})) \cong T(\tilde{H}_*(S^m \vee S^n)) = T(x,y)$$

as algebras $\quad |x| = m$

$|y| = n$.

② $\exists$ algebra iso

$$T(x,y) = UL\langle x,y\rangle$$

where $L\langle x,y\rangle$ = free Lie algebra on x,y

$UL\langle x,y\rangle$ = universal enveloping algebra.

Poincaré-Birkhoff-Witt $\Rightarrow$

$$UL\langle x,y\rangle \cong \bigotimes_\alpha S(x_\alpha)$$

↑ coalgebra iso

where α runs over a module basis for $L\langle x,y\rangle$

x_α corresponds a module generator

$S(x_\alpha)$ = free symmetric algebra generated by x_α.

③ Observe $S(x_\alpha) = H_*(\Omega S^{n_\alpha})$ where $n_\alpha = |x_\alpha| - 1$.

④ Want $\Omega S^{n_\alpha} \to \Omega(S^{m+1} \vee S^{n+1})$ realizing

$$S(x_\alpha) \hookrightarrow \bigotimes S(x_\alpha) \cong T(x,y).$$

α

Let $A \xrightarrow{E} \Omega\Sigma A$ be the suspension.

Let $u : S^m \xrightarrow{E} \Omega S^{m+1} \xrightarrow{\Omega \mathrm{incl}} \Omega(S^{m+1} \vee S^{n+1})$

$v : S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{\Omega \mathrm{incl}} \Omega(S^{m+1} \vee S^{n+1})$

Note: $u_*(1_m) = x$, $v_*(1_n) = y$.

Let $s_\alpha : S^{n_\alpha - 1} \longrightarrow \Omega(S^{m+1} \vee S^{n+1})$ be

the Samelson product formed from u, v corresponding to the bracket x_α.

Then x, y are primitive $\Rightarrow (s_\alpha)_*(1_{n_\alpha - 1}) = x_\alpha$.

Use the James construction to extend

$$\begin{tikzcd}
S^{n_\alpha - 1} \arrow[r, "s_\alpha"] \arrow[d] & \Omega(S^{m+1} \vee S^{n+1}) \\
\Omega S^{n_\alpha} \arrow[ur, dashed, "\bar{s}_\alpha"'] &
\end{tikzcd}$$

$\bar{s}_\alpha$ is an H-map $\Rightarrow (\bar{s}_\alpha)_*$ is an algebra map

$\Rightarrow (S_\alpha)_*$ induces the inclusion of $S(x_\alpha) = H_*(\Omega S^{n_\alpha})$ into

$$\bigotimes_\alpha S(x_\alpha) \cong H_*(\Omega(S^{m+1} \vee S^{n+1})).$$

⑤ Multiply:

$$\prod_\alpha \Omega S^{n_\alpha} \longrightarrow \Omega(S^{m+1} \vee S^{n+1})$$

is a htpy equivalence.

Conclusion: Traditional decomposition method is hard work.

Similar strategy has been used for:

- $\Omega P^{m+1}(p^r)$, odd p, $r \geq 1$.

  (Cohen-Moore-Neisendorfer).
- construct certain finite H-spaces as retracts of loop suspensions
  (Cohen-Neisendorfer)

- with addition idempotent techniques, Selick-Wu gave functorial decompositions of loop suspensions.

An alternative decomposition method

Based on the Cube Lemma.

Cube Lemma: Suppose $\exists$ htpy commutative cube

$$\begin{tikzcd}
E \arrow[rr] \arrow[dr] \arrow[dd] & & F \arrow[dr] \arrow[dd] & \\
& H \arrow[rr] \arrow[dd] & & G \arrow[dd] \\
A \arrow[rr] \arrow[dr] & & B \arrow[dr] & \\
& D \arrow[rr] & & C
\end{tikzcd}$$

where the bottom face is a htpy pushout and the 4 sides are htpy pullbacks. Then the top face is also a htpy pushout.

Ex: (Hilton-Milnor Theorem).

Start with the pushout

$$(*) \qquad \begin{tikzcd} * \arrow[r] \arrow[d] & X \arrow[d] \\ {} & {} \end{tikzcd}$$

$$Y \longrightarrow X \vee Y.$$

Form a cube by $X \vee Y \hookrightarrow X \times Y$ and taking htpy fibres from the 4 corners of ⊛ mapped into $X \times Y$.

$$\begin{tikzcd}
\Omega X * \Omega Y \arrow[rr, "a"] \arrow[dr, "b"] \arrow[dd] & & \Omega Y \arrow[dr] \arrow[dd] & \\
& \Omega X \arrow[rr] \arrow[dd] & & F \arrow[dd] \\
* \arrow[rr] \arrow[dr] & & X \arrow[dr] & \\
& Y \arrow[rr] & & X \vee Y \arrow[d] \\
& & & X \times Y
\end{tikzcd}$$

Cube Lemma $\Rightarrow$ top face is a htpy pushout.

Easy to see a, b are projections

$\Rightarrow F \simeq \Omega X * \Omega Y.$

<u>More</u>: $\Omega(X \vee Y) \simeq \Omega X \times \Omega Y \times \Omega F$

$\simeq \Omega X \times \Omega Y \times \Omega(\Omega X * \Omega Y).$

Ex: $\Omega(S^{m+1} \vee S^{n+1}) \simeq \Omega S^{m+1} \times \Omega S^{n+1} \times \Omega(\Omega S^{m+1} * \Omega S^{n+1})$

Further: $\Omega S^{m+1} * \Omega S^{n+1} \simeq \Sigma \Omega S^{m+1} \wedge \Omega S^{n+1}$

$$\Sigma \Omega S^{m+1} \simeq \bigvee_{k \geq 1} S^{km+1}$$

$\Rightarrow \Sigma \Omega S^{m+1} \wedge \Omega S^{n+1} \simeq$ wedge of spheres.

Iterate to get Hilton's Decomposition.

Advantage: - easier, less calculations

Disadvantage - got htpy type of F
- No information on $F \to X \vee Y$.

In what follows:
- develop a Cube Lemma method to work out ΩM for families of Poincaré Duality spaces

- build more properties into the method to help identify the maps.