

Loop Space Decompositions

$\Omega X = \text{Map}_*(S^1, X)$ - based loop space

$$\pi_m(\Omega X) = \pi_{m+1}(X)$$

Advantage of ΩX : it has a multiplication,
is an H-group.

Aim: Decompose ΩX in analogy to decomposing
a group.

Traditional Strategy:

① Calculate $H_*(\Omega X)$

② Decompose $H_*(\Omega X)$ as a tensor product

$$H_*(\Omega X) \cong \bigoplus_{\alpha} M_{\alpha}$$

↑
iso as coalgebras, modules.

③ Identify M_{α} as $H_*(X_{\alpha})$ for some
space X_{α} .

④ Find a map $X_2 \rightarrow \Omega X$ inducing

$$\pi_2 \hookrightarrow \bigoplus_2 \pi_2 \cong H_*(\Omega X).$$

⑤ Multiply together to get

$$\bigotimes_2 \pi_2 \rightarrow \Omega X$$

induces $\bigoplus_2 \pi_2 \rightarrow H_*(\Omega X)$ - an iso -

so \bigotimes is a htpy equivalence by
Whitehead's Theorem.

Steps 1-4 can be hard, Step 5 is easy.

Example: Hilton's Thm:

Decompose $\Omega(S^{m+1} \vee S^{n+1})$ for $m, n \geq 1$.

① Bott-Samelson \Rightarrow (field coeff)

$$H_*(\Omega(S^{m+1} \vee S^{n+1})) \cong T(\tilde{H}_*(S^m \vee S^n)) = T(x, y)$$

\uparrow
as algebras

$|x| = m$
 $|y| = n.$

② \exists algebra iso

$$T(x, y) = UL \langle x, y \rangle$$

where $L \langle x, y \rangle =$ free Lie algebra on x, y

$UL \langle x, y \rangle =$ universal enveloping algebra.

Poincaré-Birkhoff-Witt \Rightarrow

$$UL \langle x, y \rangle \cong \bigoplus_{\alpha} S(x_{\alpha})$$

↑
codgebra iso

where α runs over a module basis for $L \langle x, y \rangle$
 x_{α} corresponds a module generator

$S(x_{\alpha}) =$ free symmetric algebra
generated by x_{α} .

③ Observe $S(x_{\alpha}) = H * (\Omega S^{n_{\alpha}})$ where
 $n_{\alpha} = |x_{\alpha}| - 1$.

④ Want $\Omega S^{n_{\alpha}} \rightarrow \Omega(S^{m+1} \vee S^{n+1})$

realizing

$$S(x_{\alpha}) \hookrightarrow \bigoplus S(x_{\alpha}) \cong T(x, y).$$

Let $A \xrightarrow{E} \Omega A$ be the suspension.

$$\text{Let } u: S^m \xrightarrow{E} \Omega S^{m+1} \xrightarrow{\Omega \text{incl}} \Omega(S^{m+1} \vee S^{n+1})$$

$$v: S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{\Omega \text{incl}} \Omega(S^{m+1} \vee S^{n+1})$$

Note: $u_*(1_m) = x$, $v_*(1_n) = y$.

Let $s_2: S^{n+1} \rightarrow \Omega(S^{m+1} \vee S^{n+1})$ be

the Samelson product formed from u, v corresponding to the bracket x_2 .

Then x, y are primitive $\Rightarrow (s_2)_*(1_{n+1}) = x_2$.

Use the James construction to extend

$$\begin{array}{ccc} S^{n+1} & \xrightarrow{s_2} & \Omega(S^{m+1} \vee S^{n+1}) \\ \downarrow & \dashrightarrow & \rightarrow \\ \Omega S^{n+1} & & \bar{s}_2 \end{array}$$

\bar{s}_2 is an A -map $\Rightarrow (\bar{s}_2)_*$ is an algebra map

$\Rightarrow (S_2)_*$ induces the
inclusion of $S(x_2) = H_*(\Omega S^2)$
into

$$\bigoplus_{\alpha} S(x_2) \cong H_*(\Omega(S^{m+1} \vee S^n)).$$

⑤ Multiply:

$$\prod_{\alpha} \Omega S^{n_{\alpha}} \rightarrow \Omega(S^{m+1} \vee S^n)$$

is a htpy equivalence.

Conclusion: Traditional decomposition method
is hard work.

Similar strategy has been used for:

- $\Omega P^{m+r}(p^r)$, odd p , $r \geq 1$.

(Cohen-Moore-Neisendorfer).

- construct certain finite H-spaces as
retracts of loop suspensions
(Cohen-Neisendorfer)

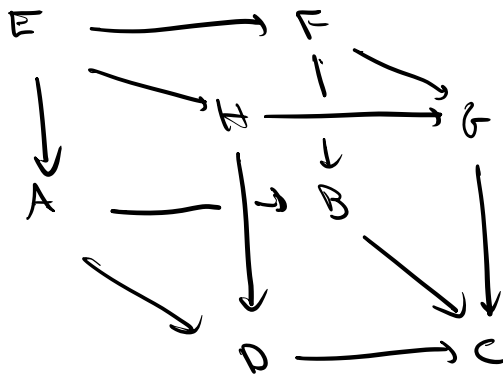
- with additional idempotent techniques,
Sickel-Wu gave functorial decompositions

of loop suspensions.

An alternative decomposition method

Based on the Cube Lemma.

Cube Lemma: Suppose \exists htpy commutative cube



where the bottom face is a htpy pushout and the 4 sides are htpy pullbacks.

Then the top face is also a htpy pushout.

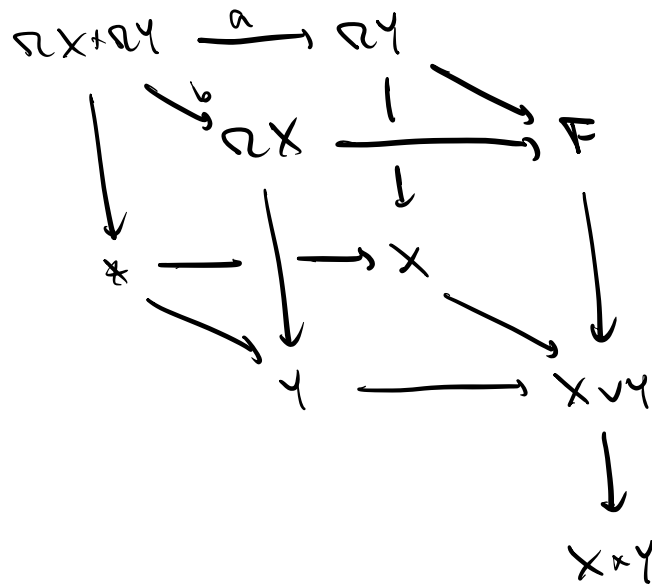
Ex: (Hilton-Milnor Theorem).

Start with the pushout

$$\begin{array}{ccc} * & \longrightarrow & X \\ \otimes & \downarrow & \downarrow \\ & 1 & 1 \end{array}$$

$$\begin{array}{ccc} & \downarrow & \downarrow \\ & Y & \rightarrow X \vee Y. \end{array}$$

Form a cube by $X \vee Y \hookrightarrow X + Y$ and taking htpy fibres from the 4 corners of \otimes mapped into $X + Y$.



Cube Lemma \Rightarrow top face is a htpy pushout.

Easy to see a, b are projections

$$\Rightarrow F \simeq \Omega X * \Omega Y.$$

More: $\Omega(X \vee Y) \simeq \Omega X * \Omega Y * \Omega F$
 $\simeq \Omega X * \Omega Y * \Omega(\Omega X * \Omega Y).$

Ex: $\Omega(S^{m+1} \vee S^{n+1}) \simeq \Omega S^{m+1} * \Omega S^{n+1} * \Omega(\Omega S^{m+1} * \Omega S^{n+1})$

Further: $\Omega S^{m+1} * \Omega S^{n+1} \cong \Sigma \Omega S^{m+1} \wedge \Omega S^{n+1}$

$$\Sigma \Omega S^{m+1} \cong \bigvee_{k \geq 1} S^{km+1}$$

$\Rightarrow \Sigma \Omega S^{m+1} \wedge \Omega S^{n+1} \cong$ wedge of spheres.

Iterate to get Hilton's Decomposition.

Advantage: - easier, less calculations

Disadvantage - got hairy type of F
- No information on $F \rightarrow XUY$.

In what follows:

- develop a cube lemma method to work out ΩM for families of Poincaré Duality spaces

- build more properties into the method to help identify the maps.