Additional Results and Open Problems

① The PD complexes considered satisfied htpy cofibrations

$$S^{n-1} \to S^m \vee S^{n-m} \vee J \to M$$

$$J \to M \to Q$$

where $H^*(Q) \cong H^*(S^m \times S^{n-m})$.

- all cases have integral classes in the "middle" cohomology dimensions.

Q: What about a PD complex given by a htpy cofibration

$$S^{n-1} \to M_{n-1} \to M$$

where $\tilde{H}_*(M_{n-1})$ is torsion?

Some cases are known.

Ⓐ If M is a $(2n-2)$-connected $(4n-1)$-dimensional PD complex with $\tilde{H}_*(M_{4n-2})$ torsion, $n \geq 2$

∃ htpy cofibration

$$S^{4n-2} \xrightarrow{f} \bigvee_{i=1}^{t} \underbrace{P^{2n}(p_i^{r_i})}_{\text{Moore space}} \to M$$

Beben-Wu, Huang-T showed:

- if each p_i is odd then ∃ htpy cofibration

$$\Sigma A \xrightarrow{f} M \xrightarrow{h} V$$

where $H^*(V) = \Lambda(x_{2n-1}, y_{2n})$

$$V = P^{2n}(m) \cup e^{4n-1}.$$

- V is the analogue of Q in the earlier case.

- $\Omega V \simeq \prod_{i=1}^{m} S^{2n-1}\{p_i^{r_i}\} \times \Omega S^{4n-1}$.

where $m = p_1^{r_1} \cdots p_\alpha^{r_\alpha}$

and $S^{2n-1}\{p_i^{r}\}$ = htpy fibre of $S^{2n-1} \xrightarrow{p^r} S^{2n-1}$

- Ωh has a right htpy inverse, by Cor C

$\exists$ htpy fibration

$$\Omega V \ltimes \Sigma A \longrightarrow M \xrightarrow{h} V$$

and $\Omega M \simeq \Omega V \times \Omega(\underbrace{\Omega V \ltimes \Sigma A}_{\text{wedge of Moore spaces.}})$.

- Moore's Conjecture holds: M is elliptic and has an exponent at every prime.

<u>Remark</u>: If M is a $(2n-1)$-connected $(4n+1)$-dimensional PD complex with $\tilde{H}_*(M_{4n})$ torsion then $\nexists$ analogue of V. So a different approach is needed.

(B) If M is an $(n-1)$-connected $(2n+1)$-dimensional PD complex with $\tilde{H}_*(M_{2n})$ torsion then $\exists$ htpy cofibration

$$S^{2n} \xrightarrow{f} \bigvee_{i=1}^{t} P^{2n+1}(p_i^{r_i}) \longrightarrow M$$

- assume ~~each~~ p_i is odd
- assume f is a sum of Whitehead products and maps that pinch trivially

to a fixed wedge summand $P^{2n+1}(p_i^{r_i})$ in $\bigvee_{i=1}^{t} P^{2n+1}(p_i^{r_i})$.

"Data"

$$\begin{tikzcd}
 & E \arrow[r] \arrow[d] & E' \arrow[d] \\
S^{2n} \arrow[r, "f"] & \bigvee_{i=1}^{t} P^{2n+1}(p_i^{r_i}) \arrow[r] \arrow[d, "\text{pinch}"] & M \arrow[d] \\
 & P^{2n+1}(p_i^{r_i}) \arrow[r, equal] & P^{2n+1}(p_i^{r_i})
\end{tikzcd}$$

Apply Theorem A:

$$\Omega M \simeq \Omega P^{2n+1}(p_i^{r_i}) \times \Omega(S^{2n+1} \vee W)$$

where W is a wedge of Moore spaces.

Case (B) may not deal with all $(n-1)$-connected $(2n+1)$-dim PD complexes. It does cover the 1-connected 5-dim PD complex.

<u>Open Problem</u>: Find a decomposition for all $(n-1)$-connected $(2n+1)$-dim PD complexes M with $\tilde{H}_*(M_{2n})$ ~~is~~ torsion. (odd primary torsion).

Open Problem: $p = 2$?!

② ∂-cones

A ∂-cone is the htpy cofibre of a map

$$\Sigma A \xrightarrow{f} \Sigma B. \quad \left(\text{or } A' \xrightarrow{f} B' \text{ for } A', B' \text{ co-}H\text{-spaces}\right).$$

Ex: Consider $\Sigma X \overset{i_1}{\hookrightarrow} \Sigma X \vee \Sigma Y$ - the inclusions.

$$\Sigma Y \overset{i_2}{\hookrightarrow} \Sigma X \vee \Sigma Y$$

Let $ad^n(i_1)(i_2)$ = the Whitehead product

$$[\underbrace{i_1,[\cdots[i_1,[i_1}_{n \text{ copies of } i_1},i_2]]\cdots]]$$

Define M_n by the htpy cofibration

$$\Sigma X^{\wedge n} \wedge Y \xrightarrow{ad^n(i_1)(i_2)} \Sigma X \vee \Sigma Y \longrightarrow M_n.$$

Define γ_n by

$$\gamma_n : \bigvee_{i=0}^{n-1} \Sigma X^{\wedge i} \wedge Y \xrightarrow{\bigvee_{t=0}^{n-1} ad^t(i_1)(i_2)} \Sigma X \vee \Sigma Y \longrightarrow M_n.$$

where $ad^0(i_1)(i_2) = i_2$

Note $\exists$ extension

$$\begin{tikzcd} \Sigma X^{\wedge n}\wedge Y \arrow[r, "ad^n(i_1)(i_2)"] & \Sigma X\vee\Sigma Y \arrow[r] \arrow[d] & M_n \arrow[d] \\ & \Sigma X \arrow[r, equal] & \Sigma X \end{tikzcd}$$

By Thm A, $\exists$ htpy fibration

$$\bigvee_{i=0}^{n-1}\Sigma X^{\wedge i}\wedge Y \xrightarrow{\gamma_n} M_n \longrightarrow \Sigma X$$

and $$\Omega M_n \simeq \Omega\Sigma X \times \Omega\Big(\bigvee_{i=0}^{n-1}\Sigma X^{\wedge i}\wedge Y\Big).$$

– if X, Y are spheres then

$$\Omega M_n \simeq \Omega\Sigma X \times \Omega W$$

where W is a wedge of spheres.

$\Rightarrow$ M_n is hyperbolic, no exponent at any prime p, and is mod-p^r hyperbolic $\forall p, \forall r \geq 1$.

<u>Open Question</u>: Find other families of spaces satisfying Moore's Conjecture and which are mod-p^r hyperbolic $\forall p, \forall r \geq 1$.

③ Rational Connections

We saw the rational notion of an inert map had a useful integral analogue.

We saw there is an integral approach using decomposition methods to proving some cases of the Gromov-Vigué-Poirrier Conjecture on exponential growth in $H^*(LX;\mathbb{Q})$.

This suggests there are more connections.

Open Question: Find more connections between rational and non-rational htpy theory.