Loop Space Decompositions in Homotopy Theory

with Applications to Poincaré Duality Spaces

Grand Objectives

X - 1-connected finite CW-complex

Goal: Calculate $\pi_*(X)$

or $[Y, X]$

Hard: $\pi_*(S^n)$ calculated through a range

Instead look at bigger pictures:

Special cases: - rational htpy groups

- v_1-periodic htpy gps
- stable range, metastable range

Global properties of htpy gps

- exponent of $\pi_*(X)$
- growth of $\mathbb{Q}$-gps in $\pi_*(X)$
- growth of torsion gps in $\pi_*(X)$

Rational vs Torsion Htpy Gps

Def: A 1-connected finite CW-complex is <u>elliptic</u> if it has finitely many rational htpy gps, it's <u>hyperbolic</u> if it has infinitely many.

<u>Rational Dichotomy</u> (Félix - Halperin - Lemaire)

If X is hyperbolic then $\bigoplus_{i \leq m} \pi_i(X) \otimes \mathbb{Q}$

grows exponentially.

Ex:
$$\left.\begin{array}{ll} \pi_*(S^{2n+1}) \otimes \mathbb{Q} & - \text{ 1 generator} \\ \pi_*(S^{2n}) \otimes \mathbb{Q} & - \text{ 2 generators} \\ \pi_*(SU(n)) \otimes \mathbb{Q} & - \text{ } n-1 \text{ generators} \end{array}\right\} \text{elliptic}$$

Ex: $\pi_*(S^n \vee S^n)$ is hyperbolic.

Def: Let p be a prime. A 1-connected finite CW-complex has <u>exponent</u> p^r if p^r is the least power of p that annihilates the p-torsion in $\pi_*(X)$.
- write as $\exp_p(X) = p^r$, or $\exp(X) = p^r$.

Ex: p odd $\exp_p(S^{2n+1}) = p^n$

— Cohen - Moore - Neisendorfer

$p = 2$ $\quad \exp_2(S^{2n+1}) \leq 2^{3/2 n + \varepsilon}$, $\varepsilon = 0, 1$

Selick

<u>Conj</u>: $\exp_2(S^{2n+1}) = 2^{n+\varepsilon}$

<u>Ex</u>: p any prime, $\exp_p(S^m \vee S^n) = \infty$

<u>Moore's Conjecture</u> - links rational and torsion htpy gps.

Let X be a 1-connected finite CW-complex. Then the following are equivalent:

1. X is elliptic
2. $\exp_p(X) < \infty$ for all primes p.
3. $\exp_p(X) < \infty$ for some prime p.

Known to hold for:
- S^n
- $S^m \vee S^n$

- finite H-spaces (elliptic) - Long.
- H-spaces with finitely-generated homology (Chacholski, Pitsch, Stanley, Scherer) - elliptic
- torsion free suspensions (Selick) - mostly hyperbolic
- polyhedral products $(D^n, S^{n-1})^K$

Partial results:

- Anick showed 2-cones satisfy Moore's Conjecture for all but finitely many p.
- McGibbon-Wilkerson: if X is elliptic then X has finite exponent for all but finitely many primes.

<u>Growth in Htpy gps</u>

<u>Def</u> (Huang-Wu) A 1-connected finite CW-complex is <u>p-hyperbolic</u> if

the p-torsion $\bigoplus_{i \leq m} \pi_i(X)$ grows exponentially with m. For a given

$r \in \mathbb{N}$, X is <u>mod-p^r hyperbolic</u> if

the number of $\mathbb{Z}/p^r$-summands in

$\bigoplus_{i \leq m} \pi_i(X)$ grows exponentially with m.

<u>Known</u>:

- $S^m \vee S^n$ - mod p^r hyperbolic $\forall p, \forall r \geq 1$.

(Boyde)

- mod p^r Moore space

Cofibration: $S^{m-1} \xrightarrow{p^r} S^{m-1} \longrightarrow P^m(p^r)$

(p odd) $P^m(p^r)$ is mod-p^t hyperbolic if

$t = r, r+1$ (Huang-Wu)
$1 \leq t < r$ (Boyde)

- Boyde has criteria for identifying spaces that are mod-p^r hyperbolic
  K-theory criterion for p-hyperbolicity
  H_*-criterion for ~~mod~~ p^r-hyperbolicity.

This suggests:

<u>Conjecture</u>: Let X be a 1-connected finite CW-complex. If X is rationally hyperbolic then it is mod p^r hyperbolic $\forall p, \forall r \geq 1$.

<u>Question</u>: Is S^n mod-p^r hyperbolic?

<u>Question</u>: Is there a 1-connected finite CW-complex that is elliptic and has p-torsion growing slower than exponentially?

Refinement of Moore's Conjecture: Let X be a 1-connected finite CW-complex. Then X is hyperbolic iff it is mod-p^r hyperbolic $\forall p, \forall r \geq 1$.

<u>Growth in the homology of free loop spaces</u>

$LX = \mathrm{Map}(S^1, X)$ - free loop space.

<u>Conjecture</u> (Gromov): If X is a 1-connected closed manifold then $H_*(LX; \mathbb{Q})$ almost always grows exponentially.

Ex: S^n needs to be excluded.

Conjecture (Vigué-Poirrier): Let X be a 1-connected finite CW-complex. If X is rationally hyperbolic then $H_*(LX;\mathbb{Q})$ grows exponentially.

Known for: - $\vee S^{n_i}$ (Vigué-Poirrier)

- $M \# N$, for M, N are manifolds not equal to spheres. (Lambrechts)

- X is coformal (Lambrechts)

There's also a connection to torsion htpy gps.

Thm (Huang-T) Let $\Sigma A \to Y \xrightarrow{h} Z$ be a htpy cofibration where A, Z are not rationally contractible. If Ωh has a right htpy inverse then:

- $H_*(LY;\mathbb{Q})$ grows exponentially

- Y is rationally hyperbolic

- Y is mod-p^r hyperbolic $\forall r \geq 1$ for all but finitely many primes.

<u>Ex</u>: 1-connected 4-manifolds.

$$S^3 \longrightarrow \bigvee_{i=1}^{d} S^2 \xrightarrow{h} M$$

<u>or</u>:

$$\bigvee_{i=1}^{d-2} S^2 \hookrightarrow M \xrightarrow{h'} Q \qquad (d \geq 2)$$

$$H^*(Q) = H^*(S^2 \times S^2)$$