Fluctuation-dissipation relations for reversible diffusions in a random environment

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Abstract

This note is based on KTGU Special lectures given by Pierre Mathieu at Kyoto University from November 10 to December 1, 2017.

1 Lecture 1: Central limit theorems.

Throughout this note, we consider the solution of the following SDE with random coefficients.

$$\begin{aligned} dX_x^\omega(t) &= \sigma^\omega(X_x^\omega(t))dW_t + b^\omega(X_x^\omega(t))dt, \\ X_x^\omega(0) &= x, \end{aligned} \tag{1}$$

where W_t is d-dim BM and σ^ω and b^ω are a random $(d \times d)$-matrix and a $\mathbb{R}^d$-valued random variable respectively. We will assume several conditions on σ^ω and b^ω. Let Ω be the collection of symmetric non-negative $(d \times d)$-matrices defined on $\mathbb{R}^d$. We will equip Ω with the topology of uniform convergence on compact subsets of $\mathbb{R}^d$. The space $\mathbb{R}^d$ naturally acts on Ω by additive translation. We denote this action by $(x, \omega) \mapsto x.\omega$. Denote by $\mathbb{Q}$ the distribution of σ^ω and b^ω.

(A1) The probability measure $\mathbb{Q}$ is invariant and ergodic with respect to the action of $\mathbb{R}^d$ on Ω. Moreover, σ^ω and b^ω satisfy the following reversibility condition: $b^\omega = \frac{1}{2} div(a^\omega)$, where $a^\omega = (\sigma^\omega)^2$.

(A2) The map $x \mapsto \sigma^\omega(x)$ is smooth for any $\omega \in \Omega$.

(A3) The matrix a^ω is uniformly elliptic in the following sense: there exists a constant $0 < \kappa < 1$ such that

$$\kappa|\xi|^2 \le |\sigma^\omega(x)\xi|^2 \le \kappa^{-1}|\xi|^2,$$

for any $\xi, x \in \mathbb{R}^d$ and any $\omega \in \Omega$.

The main purpose of this note is to study a perturbed version of the SDE (1), which is given as follows: Let $\lambda > 0$ and $e_1 \in \mathbb{R}^d$ with $|e_1| = 1$.

$$\begin{aligned} dX_x^{\lambda,\omega}(t) &= \sigma^\omega(X_x^{\lambda,\omega}(t))dW_t + b^\omega(X_x^{\lambda,\omega}(t))dt + \lambda a^\omega(X_x^{\lambda,\omega}(t)) \cdot e_1 dt, \\ X_x^{\lambda,\omega}(0) &= x. \end{aligned} \tag{2}$$

We aim to describe possible scaling limits of solutions of the SDE (2). The following results are plausible guess for an answer of this problem.

- When $\lambda = 0$, $X_x^\omega(t)$ is of order $\sqrt{t}$ and the invariance principle holds.
- When $\lambda > 0$, $X_x^{\lambda,\omega}(t)$ is of order λt (ballistic) and the law of large number holds.

Hence, an interesting problem is to study what happens when we scale λ and t at the same time in such a way that $\lambda^2 t$ tends to a positive constant, which is equivalent to $\sqrt{t} \sim c\lambda t$ for some $c > 0$. To do so, we will assume the following condition, called *finite range of dependence*, besides (A1)$\sim$(A3). For a Borel subset $F \subseteq \mathbb{R}^d$, we define the σ-field $\mathcal{H}_F$ as the one generated by $\{\sigma^\omega(x)\}_{x\in F}$.

(A4) There exists a constant $R > 0$ such that for any Borel subsets $F, G \subseteq \mathbb{R}^d$ with $\inf\{|x - y| : x \in F, y \in G\} > R$, $\mathcal{H}_F$ and $\mathcal{H}_G$ are independent.

In what follows, we will assume (A1)$\sim$(A4) unless otherwise stated.

Notation 1.1. *For fixed $\omega \in \Omega$, we denote by P^ω and E^ω the quenched law and the quenched expectation with respect to W, respectively. We denote by $\mathbb{P}$ and $\mathbb{E}$ the annealed law and the annealed expectation respectively.*

We start with introducing the following result.

Theorem 1.2 ([PV, O, KV])**.** *Under the annealed measure $\mathbb{P}$, the law of $\varepsilon X_0^\omega(\cdot/\varepsilon^2)$ converges to the law of d-dim BM with some covariance matrix Σ.*

In what follows, we take the point of view of the *environment seen from particle*, which is defined as follows. Define an Ω-valued process $\omega(t) := X_0^\omega(t).\omega$. It is not difficult to see that

- $\omega(t)$ is a Markov process under $\mathbb{P}$,
- $\mathbb{Q}$ is invariant measure of $\omega(t)$, and
- (A3) implies that $\mathbb{Q}$ is ergodic.

Definition 1.3. *Let $D = (D_1, ..., D_d)$ be the generator of the $\mathbb{R}^d$-action on Ω. Define $\sigma(\omega) := \sigma^\omega(0)$ and $b(\omega) := b^\omega(0)$. Note that we have $\sigma^\omega(x) = \sigma(x.\omega)$, $b^\omega(x) = b(x.\omega)$ and $b = \frac{1}{2}div_\Omega(a)$, where div_Ω is divergence in Ω. In other words, div_Ω is the adjoint of D .*

By using notation defined above, we can rewrite the SDE (1) as follows.

$$dX_0^\omega(t) = \sigma(\omega(t))dW_t + b(\omega(t))dt.$$

In particular, the above formula shows that X_0^ω is an *additive functional* of $\omega(\cdot)$.

1.1 H^{-1} condition

We first introduce the Dirichlet form $(\mathcal{E}, \mathcal{D})$ which is defined as follows.

$$\mathcal{E}(f,g) := \frac{1}{2}\int_\Omega \sigma Df \cdot \sigma Dg d\mathbb{Q},$$
$$\mathcal{D} := \{f \in L^2(\Omega,\mathbb{Q}) \ : \ Df \in (L^2(\Omega,\mathbb{Q}))^d\}.$$

In the sequel, we will need the following function spaces consisting of centered functions.

$$L_0^2 := \{f \in L^2(\Omega, \mathbb{Q}) \ : \ \int f d\mathbb{Q} = 0\}, \quad \mathcal{D}_0 := \{f \in \mathcal{D} \ : \ \int f d\mathbb{Q} = 0\}.$$

For $f : \Omega \mapsto \mathbb{R}$, we define the additive functional $A_f(t) := \int_0^t f(\omega(s))ds$. Next, we explain a computation which suggests an introduction of H^{-1}-condition. For $t > 0$,

$$\begin{aligned}
\frac{1}{t}\mathbb{E}[A_f(t)^2] &= \frac{2}{t}\int_0^t du \int_0^u \mathbb{E}[f(\omega(s))f(\omega(u))] \\
&= \frac{2}{t}\int_0^t du \int_0^u \mathbb{E}[f(\omega(0))f(\omega(u-s))] \quad (\because \text{by stationarity}) \\
&= 2\int_0^t du(1-\frac{u}{t})\mathbb{E}[f(\omega(0))f(\omega(u-s))].
\end{aligned}$$

Since the reversibility of $\omega(\cdot)$ implies $\mathbb{E}[f(\omega(0))f(\omega(u-s))] \geq 0$, we have

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}[A_f(t)^2] = 2\int_0^\infty \mathbb{E}[f(\omega(0))f(\omega(u))]. \tag{3}$$

Thus, we get the following equivalence between the convergence of $\frac{1}{t}\mathbb{E}[A_f(t)^2]$ and H^{-1} condition, which will be introduced again later.

$$\lim_{t\to\infty} \frac{1}{t}\mathbb{E}[A_f(t)^2] < \infty \Leftrightarrow \ H^{-1}\text{-condition} \ : \int_0^\infty du\mathbb{E}[f(\omega(0))f(\omega(u))] < \infty.$$

By applying the formula (3), we obtain that there exists a symmetric matrix Σ such that

$$\lim_{t\to\infty} t^{-1}\mathbb{E}[(X_0^\omega(t)\cdot e)^2] = e\cdot(\Sigma e). \tag{4}$$

Now, we introduce the H^{-1}-condition. Let $L := \frac{1}{2}div_\Omega(aD)$ be the generator of $\omega(\cdot)$. For $f \in L_0^2(\Omega, \mathbb{Q})$, we say that f satisfies H^{-1}-condition if either of the following equivalent statements holds.

- $\int_0^\infty du\mathbb{E}[f(\omega(0))f(\omega(u))] < \infty$,
- f is in the domain of $(-L)^{1/2}$,
- $\sup_{g\in\mathcal{D}_0} |\int fgd\mathbb{Q}|^2/\mathcal{E}(g,g) < \infty$.

We next define H^1 space, and then define H^{-1} space as its dual. Note that $\mathcal{E}$ on $\mathcal{D}_0$ is a norm due to ergodicity. Define H^1 space as the completion of $\mathcal{D}_0$ with respect to $\mathcal{E}$, and H^{-1} space as its dual space. Then, for $f \in H^{-1}$, the H^{-1} norm $\|f\|_{H^{-1}}$ is given as follows.

$$\|f\|_{H^{-1}}^2 := \int_0^\infty du\mathbb{E}[f(\omega(0))f(\omega(u))] = \int(-L)^{-1}f(\omega)\cdot f(\omega)d\mathbb{Q}(\omega).$$

Theorem 1.4 (Kip-Var)**.** *For $f \in L_0^2(\Omega,\mathbb{Q})\cap H^{-1}$, under $\mathbb{P}$, the law of $\varepsilon A_f(\cdot/\varepsilon^2)$ converges to a 1-dim BM with variance $\Sigma(f) := \|f\|_{H^{-1}}^2$.*

Lemma 1.5. *For $f \in L_0^2(\Omega, \mathbb{Q}) \cap H^{-1}$, we have*

$$\mathbb{E}\left[\sup_{s\le t}(A_f(s))^2\right] \le 8t\|f\|_{H^{-1}}, \quad \textit{for all } t > 0.$$

See [MP, Lemma 2.2] for the proof. By the above result, we can define A_f for all $f \in H^{-1}$. We explained the following convergence so far.

$$\begin{aligned}
&\varepsilon X_0^\omega(\cdot/\varepsilon^2) \xrightarrow{\varepsilon\to 0} d\text{-dim } BM(\Sigma),\\
&\varepsilon A_f(\cdot/\varepsilon^2) \xrightarrow{\varepsilon\to 0} 1\text{-dim } BM(\Sigma(f)).
\end{aligned}$$

Moreover, we can obtain the following joint convergence because both of the above convergence is based on martingale approximation.

$$\begin{pmatrix} \varepsilon X_0^\omega(\cdot/\varepsilon^2) \\ \varepsilon A_f(\cdot/\varepsilon^2) \end{pmatrix} \xrightarrow{\varepsilon\to 0} (d+1)\text{-dim BM with covariance matrix } \begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma(f) \end{pmatrix}.$$

Note that the cross product of $\varepsilon X_0^\omega(\cdot/\varepsilon^2)$ and $\varepsilon A_f(\cdot/\varepsilon^2)$ is zero since X_0^ω is an antisymmetric additive functional and A_f is a symmetric additive functional.

1.2 Study of the perturbed SDE (2)

In this subsection, we will explain that things discussed in the previous subsection can be used to study the perturbed SDE (2) via the *Girsanov transform*. We let t tend to ∞ and λ tend to 0 in such a way that $\lambda^2 t \to \alpha$ for some $\alpha > 0$. Recall that $X_0^{\lambda,\omega}$ is the solution of the following SDE.

$$\begin{aligned}
&dX_0^{\lambda,\omega}(t) = \sigma^\omega(X_0^{\lambda,\omega}(t))dW_t + b^\omega(X_0^{\lambda,\omega}(t))dt + \lambda a^\omega(X_0^{\lambda,\omega}(t)) \cdot e_1 dt,\\
&X_0^{\lambda,\omega}(0) = 0.
\end{aligned}$$

By the Girsanov formula, we have the following. For a bounded measurable function F,

$$E^\omega\left[F(X_0^{\lambda,\omega}(s)\ ;\ s \le t)\right] = E^\omega\left[F(X_0^\omega(s)\ ;\ s\le t)\exp\left(\lambda\overline{B}(t) - \frac{\lambda^2}{2}\langle\overline{B}\rangle(t)\right)\right],$$

where

$$\overline{B}(t) := \int_0^t \sigma^\omega(X_0^\omega(s))\cdot e_1 dW_s = X_0^\omega(t) - \int_0^t b^\omega(X_0^\omega(s))ds. \tag{5}$$

Thus, we get

$$\begin{aligned}
&\mathbb{E}\left[F(\varepsilon X_0^{\lambda,\omega}(s/\varepsilon^2)\ ;\ 0\le s\le t)\right]\\
=&\mathbb{E}\left[F(\varepsilon X_0^\omega(s/\varepsilon^2)\ ;\ 0\le s\le t)\exp\left(\lambda\overline{B}(t/\varepsilon^2) - \frac{\lambda^2}{2}\langle\overline{B}\rangle(t/\varepsilon^2)\right)\right].
\end{aligned}$$

Discussions in the previous subsection implies the invariance principle for $(X_0^\omega, \overline{B})$. Applying the invariance principle for $(X_0^\omega, \overline{B})$ to the above equality, the limit of the RHS of the above equality turns out to be

$$\mathbb{E}\left[F(Z_1(s)\ ;\ 0\le s\le t)\exp\left(\sqrt{\alpha}Z_2(t) - \frac{\alpha}{2}\langle Z_2\rangle(t)\right)\right],$$

where (Z_1, Z_2) is a $(d+1)$-dim BM whose covariance matrix is given as follows.

$$\begin{pmatrix} \Sigma & \Sigma e_1 \\ (\Sigma e_1)^{\mathrm{T}} & \text{something irrelevant} \end{pmatrix}.$$

Note that we use orthogonality of symmetric and antisymmetric additive functionals and the second equality of the formula (5) for the computation of the cross product. On the other hand, we the following by the Girsanov formula.

$$\begin{aligned} &\mathbb{E}\left[F(Z_1(s)\ ;\ 0 \le s \le t)\exp\left(\sqrt{\alpha}Z_2(t) - \frac{\alpha}{2}\langle Z_2\rangle(t)\right)\right] \\ =&\mathbb{E}\left[F(Z_1(s) + \mu s\ ;\ 0 \le s \le t)\right], \end{aligned}$$

where $\mu = \sqrt{\alpha}\Sigma e_1$. This is the result obtained in [LR].

Theorem 1.6 ([LR])**.** *Let $\alpha > 0$. As λ and ε tend to 0 in such a way that λ^2/ε^2 tends to α, under $\mathbb{P}$, the law of $\varepsilon X_0^{\lambda,\omega}(\cdot/\varepsilon^2)$ converges to the law of a d-dim BM with constant drift whose covariance matrix and drift term are Σ and $\sqrt{\alpha}\Sigma e_1$, respectively.*

Let $f \in H^{-1}$. For the same reason we discuss below Lemma 1.5, we obtain that under $\mathbb{P}$, the law of $(\varepsilon X_0^{\lambda,\omega}(\cdot/\varepsilon^2), \varepsilon A_f(\cdot/\varepsilon^2))$ converges to the law of $(d+1)$-dim BM with constant drift whose covariance matrix and drift are

$$\begin{pmatrix} \Sigma & 0 \\ 0 & \Sigma(f) \end{pmatrix}, \quad \begin{pmatrix} \Sigma e_1 \\ \overline{\Gamma}(f) \end{pmatrix}, \quad \text{respectively.}$$

We will compute $\overline{\Gamma}(f)$.

$$\begin{aligned} \overline{\Gamma}(f) &= \lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[A_f(t)\overline{B}(t)\right] = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[A_f(t)(-A_{b\cdot e_1}(t))\right] \\ &= -2\langle f, b\cdot e_1\rangle_{H^{-1}} = -\Sigma(f, b\cdot e_1), \end{aligned}$$

where $\Sigma(f,g) := 2\langle f, g\rangle_{H^{-1}}$. Recall that $\Sigma(f) = 2\|f\|_{H^{-1}}^2$.

2 Lecture 2: Fluctuation-dissipation relations.

In this section, we will introduce the notion of *steady state*, which is an invariant measure for the Markov process $\omega^\lambda(\cdot)$, and study a couple of its properties. Recall that the generators of the Markov processes $X_x^\omega(t)$ and $X_x^{\lambda,\omega}(t)$ are respectively given by

$$\begin{aligned} \mathcal{L}^\omega &= \frac{1}{2}div(a^\omega\nabla), \\ \mathcal{L}^{\lambda,\omega} &= \frac{1}{2}e^{\lambda V}div(e^{-\lambda V}a^\omega\nabla), \end{aligned}$$

where $V(y) = -2y\cdot e_1$. We next give the definition of steady state. Note that for $\lambda > 0$, the probability measure $\mathbb{Q}$ is no longer invariant.

Definition 2.1. *For $\lambda > 0$, a Borel probability measure ν_λ on Ω is said to be steady state if for any bounded continuous function f, for $\mathbb{Q}$-a.e. ω and P almost surely, we have*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\omega^\lambda(s))ds = \nu_\lambda(f),$$

where $\omega^\lambda(s) := X_0^{\lambda,\omega}(s).\omega$.

Note that steady state ν_λ is an invariant for $\omega^\lambda(\cdot)$ and unique if exists. One of the purposes of introduction of steady state is to utilize it to prove the following *Einstein relation*.

$$\lim_{\lambda\to 0} \frac{1}{\lambda} l(\lambda) = \Sigma e_1, \text{ where } l(\lambda) = \lim_{t\to\infty} \frac{X_0^{\lambda,\omega}(t)}{t} \quad \mathbb{Q}\text{-}a.e.$$

See (4) for the definition of the matrix Σ. Of course, existence of the *speed* $l(\lambda)$ is not obvious at all and we will prove its existence in Section 3 and 4 using regeneration times.

2.1 Intuition

In this subsection, we will give an intuitive explanation of things behind the proof of the Einstein relation. What we will do here is to compute $\partial_{\lambda=0}\nu_\lambda$ by using non-rigorous arguments. By the definition of steady state, we have

$$\partial_{\lambda=0}\nu_\lambda(f) = \partial_{\lambda=0} \lim_{t\to\infty} \mathbb{E}\left[\frac{1}{t}\int_0^t f(\omega^\lambda(s))ds\right].$$

If we exchange two limits in the RHS (not rigorous yet!), we get

$$\begin{aligned}
\partial_{\lambda=0}\nu_\lambda(f) &= \lim_{t\to\infty} \partial_{\lambda=0}\mathbb{E}\left[\frac{1}{t}\int_0^t f(\omega^\lambda(s))ds\right] \\
&= \lim_{t\to\infty} \partial_{\lambda=0}\mathbb{E}\left[\frac{1}{t}\int_0^t f(\omega(s))ds \cdot \exp\left(\lambda\overline{B}(t) - \frac{\lambda^2}{2}\langle\overline{B}\rangle(t)\right)\right] \\
&= \lim_{t\to\infty} \partial_{\lambda=0}\mathbb{E}\left[\frac{1}{t}\int_0^t f(\omega(s))ds \cdot \overline{B}(t)\right] = \overline{\Gamma}(f). \quad \text{(See the first lecture.)}
\end{aligned}$$

Note that we use the Girsanov formula in the second equality. But there are two technical problems in the above computations.

1. It is not clear whether we can exchange two limits.
2. $\overline{\Gamma}(f)$ is defined for $f \in H^{-1}$ though $\nu_\lambda(f)$ makes sense only for $f \in L^1(\nu_\lambda)$.

2.2 Introduction of H^{-1}_∞ and $\tilde{H}^{-1}_\infty$

Let F be a vector valued function in $(L^2(\Omega,\mathbb{Q}))^d$. Then, the formula

$$\langle F, u\rangle := -\int_\Omega F \cdot Du d\mathbb{Q}$$

defines a continuous linear functional on $H^1(\Omega)$. Thus, there exists an element $f \in H^{-1}$ such that $\langle F, u\rangle = \langle f, u\rangle_{H^{-1}}$. This implies that $f = div_\Omega F$. We now define subspaces of H^{-1}.

Definition 2.2. *Define*

$$H_\infty^{-1} := \{f \in H^{-1} \ ; \ f = div_\Omega F \text{ for some } F \in (L^\infty(\Omega, \mathbb{Q}))^d\},$$
$$\|f\|_{H_\infty^{-1}} := \min\{\|F\|_\infty \ ; \ f = div_\Omega F\}.$$

Then, it can be checked that $(H_\infty^{-1}, \|\cdot\|_{H_\infty^{-1}})$ is a Banach space. (See [MP, Section 3.1.].) Analogously, define

$$\tilde{H}_\infty^{-1} := \{f \in H_\infty^{-1} \ ; \ f = div_\Omega F \text{ for some bounded continuous function } F\}.$$

2.3 Steady state functional

Definition 2.3. *A continuous linear functional ν_λ on $\tilde{H}_\infty^{-1}$ is called steady state functional if for any $f \in \tilde{H}_\infty^{-1}$*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\omega^\lambda(s))ds = \nu_\lambda(f),$$

$\mathbb{Q}$-a.e., and in $L^1(P)$.

Our goal is to compute the derivative of steady state. To do so, we first prove the Lipschitz continuity of steady state functional.

Theorem 2.4. *Assume that there is a steady state functional ν_λ for all $0 < \lambda < 1$. Then, there exists a constant $c_1 > 0$ such that for any $f \in H_\infty^{-1}$*

$$|\nu_\lambda(f)| \le c_1\lambda\|f\|_{H_\infty^{-1}}.$$

3 Lecture 3: A priori estimates on diffusions.

We see in Theorem 2.4 that $X_0^{\lambda,\omega} \cdot e_1 \le c\lambda t$ for some $c > 0$. One of aims in this section is to prove the lower bound. Specifically, we will prove the following.

$$X_0^{\lambda,\omega} \cdot e_1 \ge c\lambda t, \text{ for } \lambda^2 t \ge 1.$$

The reason why we need $\lambda^2 t \ge 1$ is that for the process to be ballistic, the effect of the drift term should be strong enough compared to the effect of the diffusive part. We will assume $e_1 = e := (1, 0, ..., 0)$ without loss of generality. The following estimate plays an important role in what follows.

Notation 3.1. *Let $L > 0$. Define $\Pi := \{x \in \mathbb{R}^d \ ; \ -L \le x \cdot e \le L\}$. For $x \in \mathbb{R}^d$, define*

$$u(x) := P^\omega\left(X_x^{\lambda,\omega} \text{ exits } \Pi \text{ from the right side}\right).$$

Define $T_{\pm L}$ to be the first hitting time of $\{x \cdot e = \pm L\}$ by the $X_0^{\lambda,\omega}$ respectively.

Lemma 3.2. *There exists L_0 depending only on κ and d such that for all $L \ge L_0$, $u(0) \ge 2/3$.*

We will apply this lemma to the rescaled process $\tilde{X}_0^{\lambda,\omega}(t) := \lambda X_0^{\lambda,\omega}(t/\lambda^2)$. The following results are consequences of Lemma 3.2.

Lemma 3.3. *There exists $c, \kappa_1 > 0$ depending only on κ, d such that*

$$P^\omega(T_{-L} < +\infty) \le ce^{-\kappa_1 \lambda L},$$

for all $\omega \in \Omega$, all $L > 0$ and all $\lambda < 1$.

Proof. We only need to combine Lemma 3.2 with a suitable coupling with a biased RW on $\mathbb{Z}$. □

Lemma 3.4. *There exists $c, \kappa_2 > 0$ depending only on κ and d such that*

$$P^\omega(T_L \ge t) \le ce^{-\kappa_2 \lambda^2 t + \lambda L},$$

for all $\omega \in \Omega$, all $L > 0$, all $t > 0$ and all $\lambda < 1$.

Proof. We only need to combine Lemma 3.2 with the following Aronson bound [A]: there exists a constant $\delta_0 > 0$ such that

$$P(|X^{\lambda,\omega}(1) \cdot e| \ge L_0) \ge \delta_0.$$

□

The above estimates will be utilized in section 4. We summarize below things we want to prove.

- Existence of steady state and steady state functional.
- The computation of the derivative of steady state. Specifically,

$$\partial_{\lambda=0}\nu_\lambda(f) = \overline{\Gamma}(f), \text{ for } f \in H^{-1}.$$

  The above relation is known as the *fluctuation-dissipation theorem.*

To prove the above claims, we utilize the assumption (A4). The function $f : \Omega \mapsto \mathbb{R}$ is called *local* if there exists a constant $R_f > 0$ such that f is $\mathcal{H}_{B(x,R_f)}$-measurable for some $x \in \mathbb{R}^d$. The following theorem gives the existence of steady state.

Theorem 3.5. *For all $\lambda > 0$, there exists a unique Borel probability measure ν_λ on Ω such that for any bounded local function, we have*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\omega^\lambda(s))ds =: \nu_\lambda(f)$$

exists P-a.s., and $\mathbb{Q}$-a.s.

Theorem 3.6. *Let $f \in H_\infty^{-1}$. When f is local,*

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t f(\omega^\lambda(s))ds =: \nu_\lambda(f)$$

exists P-a.s., and $\mathbb{Q}$-a.s.

Once we have the two theorems above, we can extend the definition of $\nu_\lambda(f)$ for any bounded continuous function by using Theorem 2.4. See the proof of [MP, Corollary 4.3] for details.

The following result is called the fluctuation-dissipation theorem, which is one of main goals of this article.

Theorem 3.7. *Let $f \in H_\infty^{-1}$. When f is local,* $\lim_{\lambda\to 0} \frac{1}{\lambda}\nu_\lambda(f) = \overline{\Gamma}(f)$.

The proof of Theorem 3.7 is given in the end of this note. By combining Theorem 3.7 with the estimate given in [MP, Lemma 3.1.] and imposing a stronger condition on f, we can relax the locality assumption as follows.

Theorem 3.8. *For $f \in \tilde{H}_\infty^{-1}$, we have* $\lim_{\lambda\to 0} \frac{1}{\lambda}\nu_\lambda(f) = \overline{\Gamma}(f)$.

We will prove the above theorems by using *regeneration times*. We first introduce notation and define regeneration times in next section. For $\lambda > 0$, define the annealed measure $\mathbb{P}_x^\lambda$ on $\Omega \times C([0,\infty);\mathbb{R}^d)$ by

$$\mathbb{P}_x^\lambda(A) := \int d\mathbb{Q}(\omega) \int \mathbf{1}_{(\omega,w)\in A} dP_x^{\lambda,\omega}(w).$$

Let $X(t)$ and $Z(t)$ be the coordinate process on $C([0,\infty);\mathbb{R}^d)$ and $C([0,\infty);\mathbb{R}^{d+1})$ respectively. Fix $f \in H^{-1}$. We also denote by $\mathbb{P}_x^\lambda$ the annealed law of the process

$$Z_x^{\lambda,\omega} := (X_x^{\lambda,\omega}(t), \int_0^t f(\omega^\lambda(s))ds + W^1(t); t \geq 0),$$

where $W(t)^1$ is a 1-dim BM. Regeneration times are a increasing sequence of random times $\tau_1^\lambda < \tau_2^\lambda < ...$ such that

$$\left\{ \left(Z(\tau_{k+1}^\lambda) - Z(\tau_k^\lambda), \tau_{k+1}^\lambda - \tau_k^\lambda \right) \right\}_{k\in\mathbb{N}}$$

are *i.i.d.* random variables. Once we construct regeneration times and obtain some moment bounds for them, we can prove Theorem 3.5 by the law of large numbers. In the sequel, we will need the following moment estimates of regeneration times, which are essentially proved in [Sh, Theorem 4.9, Corollary 4.10]. See Proposition 4.9 in [MP] for details.

Proposition 3.9. *Let f be a local function in H_∞^{-1}. Then, there exists constants $C(f), \overline{C}(f) > 0$ such that*

$$\mathbb{E}^\lambda \left[\exp\left(C(f)\lambda^2 \tau_1^\lambda \right) \right] < \overline{C}(f), \text{ and } \mathbb{E}^\lambda \left[\exp\left(C(f)\lambda \max_{s\leq\tau_1^\lambda} |Z(s)| \right) \right] < \overline{C}(f).$$

4 Lecture 4: Regeneration times and steady states.

In this section, we will give a sketchy explanation of the construction of regeneration times. See [Sh, GMP, MP] for details. In what follows, we first discuss the construction of regeneration times for $X^{\lambda,\omega}$, and the briefly discuss the construction for $Z^{\lambda,\omega}$. We start with the construction of τ_1^λ. The strategy is as follows:

Step 1. Wait until first time when the trajectory of $X^{\lambda,\omega}$ has progressed by distance $2/\lambda$ from its past maximum in direction e_1. We denote such time by N^λ. Note that N^λ is a stopping time.

Step 2. Toss a coin, which is modeled by a Bernoulli random variable, and if it fails, add time $1/\lambda^2$ and then go back to Step1. Denote by N_k^λ the time when the k-th coin tossing is carried out. If it successes, add time $1/\lambda^2$ and then force the trajectory to make a deterministic jump, which is independent of ω, of size $O(1/\lambda)$ in direction e_1. Let $S_k^\lambda := N_k^\lambda + 1/\lambda^2$.

Step 3. Assume that the coin tossing successes for the first time at the j-th trial. Define

$$D^\lambda := \inf\left\{t > S_j^\lambda \ ; \ (X_0^{\lambda,\omega}(t) - X_0^{\lambda,\omega}(S_K^\lambda)) \cdot e_1 \le -\frac{1}{3\lambda}\right\}.$$

If $D^\lambda < \infty$, then go back to Step 1. If $D^\lambda = \infty$, then set $\tau_1^\lambda := S_j^\lambda$, and start again with Step 1 after time τ_1^λ to define $\tau_2^\lambda < \tau_3^\lambda < ....$ Then, the law of trajectory $X(\cdot + \tau_1^\lambda) - X(\tau_1^\lambda)$ is $\mathbb{P}^\lambda(\ \cdot\ |D^\lambda = \infty)$.

By the construction, $X(\tau_1^\lambda)$ depends only on the environment $\sigma^\omega(y)$ for $y \cdot e_1 \le X(\tau_1^\lambda) \cdot e_1 - 1/\lambda$, and the event $\{D^\lambda = \infty\}$ depend only on $\sigma^\omega(y)$ for $y \cdot e_1 \ge X(\tau_1^\lambda \cdot e_1) - 1/3\lambda$. Remark that we need to enlarge the probability space Ω to define a coin tossing. We explain a coin tossing involved in the above strategy. Let Y be a Bernoulli random variable with success probability $0 < \delta < 1$. Suppose that we now toss a coin Y at location x at time t.

When $Y = 1$ (success): Sample $X(t + 1/\lambda^2)$ with uniform law on the ball $B(x + 1/\lambda, 10^{-5}/\lambda)$.

When $Y = 0$ (fail): Sample $X(t + 1/\lambda^2)$ so that

$$\begin{aligned}\mathbb{P}(X_x^{\lambda,\omega}(1/\lambda^2) \in \cdot) =& \delta(\text{the normalized uniform law on } B(x + 1/\lambda, 10^{-5}/\lambda)) \\ &+ (1-\delta)\mathbb{P}_x^\lambda(X(1/\lambda^2) \in \cdot \mid Y = 0).\end{aligned}$$

The choice of the value δ is related to the Aronson bound. See [Sh, GMP] for details. the following moment estimates for τ_1^λ, proved in [MP, Proposition 5.3], give information about the order of magnitude of τ_1^λ.

Proposition 4.1. *There exists a constant $C_1(f) > 0$ such that for all* $0 < \lambda \le 1$

$$\mathbb{E}^\lambda\left[\exp\left(C_1(f)\lambda^2\tau_1^\lambda\right)\right] < \infty, \quad \text{and} \quad \mathbb{E}^\lambda\left[\exp\left(C_1(f)\lambda(e_1 \cdot X(\tau_1^\lambda))\right)\right] < \infty$$

4.1 The law of large numbers and the central limit theorem

For fixed λ, by regeneration times and the moment estimates in Proposition 4.1, we can decompose $X^{\lambda,\omega}$ into the sum of increments between successive regeneration times, which are $\mathbb{R}^d$-valued *i.i.d.* random variables, and a negligible error term. Thus, $X^{\lambda,\omega}/t$ converges $\mathbb{Q}$-a.s., and

P-a.s. We will denote the limiting vector by $l(\lambda)$, and call it the *speed* of $X^{\lambda,\omega}$. We have the following expression of $l(\lambda)$.

$$l(\lambda) = \frac{\mathbb{E}^{\lambda}\left[X(\tau_2^{\lambda}) - X(\tau_1^{\lambda}))\right]}{\mathbb{E}^{\lambda}\left[\tau_2^{\lambda} - \tau_1^{\lambda}\right]} = \frac{\mathbb{E}^{\lambda}\left[X(\tau_1^{\lambda}) \mid D^{\lambda} = \infty\right]}{\mathbb{E}^{\lambda}\left[\tau_1^{\lambda} \mid D^{\lambda} = \infty\right]}.$$

Remark that we can check that $l(\lambda) \cdot e_1$, which is plausible by the definition of $X^{\lambda,\omega}$. By the same reason, we also have the following central limit theorem. Under $\mathbb{P}^{\lambda}$

$$\frac{X^{\lambda,\omega}(t) \cdot e_1 - (l(\lambda) \cdot e_1)\, t}{\sqrt{t}}$$

weakly converges to the centered normal distribution with covariance $e_1 \cdot \Sigma_{\lambda} e_1$, where

$$e_1 \cdot \Sigma_{\lambda} e_1 = \frac{\mathbb{E}\left[\left(X(\tau_1^{\lambda}) \cdot e_1 - (l(\lambda) \cdot e_1)\tau_1^{\lambda}\right)^2 \mid D^{\lambda} = \infty\right]}{\mathbb{E}[\tau_1^{\lambda} \mid D^{\lambda} = \infty]}.$$

The construction of regeneration times for $Z^{\lambda,\omega}$ is similar to that for $X^{\lambda,\omega}$. One of differences is that we have to change the value of success probability δ into a positive constant depending on f that is related to the Aronson bound for generator of $Z^{\lambda,\omega}$, which is given by

$$\mathcal{M}^{\lambda,\omega} := (\mathcal{L}^{\lambda,\omega})_x + \frac{1}{2}\partial_y^2 + f(x.\omega)\partial_y,$$

for $z = (x, y),\ x \in \mathbb{R}^d,\ y \in \mathbb{R}$. Recall that

$$Z_z^{\lambda,\omega} := (X_x^{\lambda,\omega}, y + W^1(t) + \int_0^t f(\omega^{\lambda}(s))ds).$$

Thus, we can prove Theorem 3.5 and Theorem 3.6 by using regeneration times of $Z^{\lambda,\omega}$.

5 Lecture 5: FDR and scaling limits.

In this section, we will finish the proof of the Einstein relation and the fluctuation-dissipation theorem. Recall that $X^{\lambda,\omega}$ and $Z^{\lambda,\omega}$ are given by

$$\begin{aligned} &dX_0^{\lambda,\omega}(t) = \sigma^{\omega}(X_0^{\lambda,\omega}(t))dW_t + b^{\omega}(X_0^{\lambda,\omega})dt + \lambda\sigma^{\omega}(X_0^{\lambda,\omega}(t))dt, \\ &X_0^{\lambda,\omega}(0) = 0, \end{aligned}$$

and

$$Z_0^{\lambda,\omega}(t) = (X_0^{\lambda,\omega}\ ;\ W_t^1 + \int_0^t f(\omega^{\lambda}(s))ds).$$

By the regeneration times constructed in previous sections, we already know that when f is local and either bounded or in H_{∞}^{-1}, we have the following convergence $\mathbb{Q}, P$ almost surely.

$$\frac{1}{t}\int_0^t f(\omega^{\lambda}(s))ds \to \nu_{\lambda}(f) = \frac{\mathbb{E}^{\lambda}[A^f(\tau_1^{\lambda}); D^{\lambda} = \infty]}{\mathbb{E}^{\lambda}[\tau_1^{\lambda}; D^{\lambda} = \infty]}.$$

This proves the existence of steady states. Now we turn to the proof of the Einstein relation. We have

$$\left|\frac{l(\lambda)}{\lambda} - \Sigma e_1\right| \le \left|\frac{l(\lambda)}{\lambda} - \frac{\mathbb{E}[X_0^{\lambda,\omega}(t)\cdot e_1]}{\lambda t}\right| + \left|\Sigma e_1 - \frac{\mathbb{E}[X_0^{\lambda,\omega}(t)\cdot e_1]}{\lambda t}\right|.$$

By Theorem 1.6, the second term vanishes when λ tends to 0 and t tends to ∞ so that $\lambda^2 t$ tends to α for some fixed $\alpha > 0$. To end the proof, we will show the following estimate.

$$\lim_{\alpha\to\infty}\ \lim_{\lambda\to 0, t\to\infty, \lambda^2 t\to\alpha} \mathrm{Var}\left(\frac{l(\lambda)}{\lambda} - \frac{\mathbb{E}[X_0^{\lambda,\omega}(t)\cdot e_1]}{\lambda t}\right).$$

We will only give a sketch of the proof below. Note that

$$\left\{\left(Z(\tau_{k+1}^{\lambda}) - Z(\tau_k^{\lambda}), \tau_{k+1}^{\lambda} - \tau_k^{\lambda}\right)\right\}_{k\in\mathbb{N}}$$

are *i.i.d.* random variables. Thus, under the annealed measure $\mathbb{P}$, the computation of the variance of $X^{\lambda,\omega}(\tau_k^{\lambda})$ is quite elementary. Hence we have

$$\mathrm{Var}\left(\frac{l(\lambda)}{\lambda} - \frac{X^{\lambda,\omega}(\tau_{n(t)}^{\lambda})}{\lambda t}\right) \le \frac{n(t)}{\lambda^2 t^2} \sim t^{-1} \to 0,$$

where

$$n(t) := \frac{t}{\mathbb{E}^{\lambda}[\tau_1^{\lambda} \mid D^{\lambda} = \infty]}.$$

By using the convergence of variance to 0 again, we also obtain that for any $\varepsilon > 0$, we have $\tau_{n(t)}^{\lambda} \sim t + \varepsilon t$ on the event whose probability tends to 1 as $t \to \infty$. Finally, using the Aronson bound, we get the control of fluctuation of diffusion processes on time intervals of size εt. More precisely, we show that the process $X^{\lambda,\omega}$ fluctuates at most $\lambda\varepsilon t$ on such time intervals. Thus, we obtain that

$$\left|\frac{X^{\lambda,\omega}(t)}{\lambda t} - \frac{X^{\lambda,\omega}(\tau_{n(t)}^{\lambda})}{\lambda t}\right| \le \varepsilon \quad \text{on the event } \{\tau_{n(t)}^{\lambda} \sim t + \varepsilon t\}.$$

By letting $\alpha \to \infty$, we get the conclusion.

Finally, we will prove the fluctuation-dissipation theorem by a different approach. We will utilize the following result.

Lemma 5.1. *For $m \in \mathbb{N}$, let $H(m)$ be a $\mathcal{F}_{\lambda^{-2}m}$-measurable bounded random variable. Then we have*

$$\mathbb{E}^{\lambda}[H(\lambda^2\tau_1^{\lambda}) \mid D^{\lambda} = \infty] = \sum_{k\ge 1} \mathbb{E}^{\lambda}[H(\lambda^2 S_k^{\lambda})\mathbf{1}_{S_k^{\lambda} < D^{\lambda}}].$$

By using the above result, we obtain the following proposition. Recall that $\lambda Z_0^{\lambda,\omega}(\cdot/\lambda^2)$ converges in law the $(d+1)$-dim BM with constant drift with the covariance matrix

$$\begin{pmatrix} \Sigma & 0 \\ 0 & 1+\Sigma(f) \end{pmatrix},$$

and the drift

$$\begin{pmatrix} \Sigma e_1 \\ \overline{\Gamma}(f) \end{pmatrix}.$$

Denote the limiting distribution by $\mathcal{P}$ and its expectation by $\mathcal{E}$.

Proposition 5.2. *Let $\phi : \mathbb{R}^{d+1} \times \mathbb{R}_+ \times [0,1]$ be a continuous function which has at most polynomial growth. Then we have*

$$\lim_{\lambda \to 0} \frac{\mathbb{E}^\lambda[\phi(\lambda Z^{\lambda,\omega}(\tau_1^\lambda), \lambda^2 \tau_1^\lambda, \lambda) \mid D^\lambda = \infty]}{\mathbb{E}^\lambda[\lambda^2 \tau_1^\lambda \mid D^\lambda = \infty]} = \frac{[\phi(Z(\tau_1), \tau_1, 0) \mid D = \infty]}{\mathcal{E}[\tau_1 \mid D = \infty]}.$$

By the above Proposition, we obtain that for $f \in \tilde{H}_\infty^{-1}$

$$\begin{aligned} \frac{\nu_\lambda(f)}{\lambda} &= \frac{\mathbb{E}^\lambda[\lambda A^f(\tau_1^\lambda) \mid D^\lambda = \infty]}{\lambda^2 \mathbb{E}^\lambda[\tau_1^\lambda \mid D^\lambda = \infty]} \\ &= \text{the drift term the last component of } Z^{\lambda,\omega} \text{ under } \mathcal{P} = \overline{\Gamma}(f). \end{aligned}$$

This proves the fluctuation-dissipation theorem.

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