Linear and Non-linear Stochastic Process Models for Lagrangian Movements of Individual Solute Particles in 1-D Open Channels

（1 次元開水路における個々の溶質粒子の Lagrange 的運動に対する
線型・非線型の確率過程モデル）

Key words: Solute transport phenomena, Stochastic differential equations, Kolmogorov's forward equation

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1. INTRODUCTION

Assessing flows and solute transport phenomena in surface water bodies is one of the most important research topics in hydro-environmental engineering because of their close relationships with a wide variety of issues on water environment and water resources, such as eutrophication of rivers and agricultural drainage canals and hypoxia in lakes and lagoons.

Analyzing solute transport phenomena in turbulent water bodies reduces to tracking the velocities and paths of individual solute particles behaving as stochastic processes. Yoshioka *et al.*[1)] proposed a Lagrangian particle tracking model that consists of the Stochastic Differential Equations (SDEs) for the velocities and paths of individual solute particles. A linearization procedure was applied to the drift of the model for deducing a system of transport equations; however, its verification has not been made so far. The purpose of this study is verification of the linearization procedure from the viewpoint of probability density functions (PDFs) and statistical moments. For the sake of simplicity, this paper considers solute transport phenomena in 1-D open channels under longitudinally uniform flow conditions.

2. NL- AND L- SDE FOR THE VELOCITY OF OPEN CHANNEL

The nonlinear SDE (NL-SDE)[1)] is given by

$$\mathrm{d}V = \left(F - CV\left|V\right|\right)\mathrm{d}t + \sigma \mathrm{d}B \tag{1}$$

where V is the velocity of 1-D open channel, F is the deterministic force, C is the damping coefficient, t is the time, σ is the volatility modulating magnitude of stochasticity of particle movements, and B is the 1-D standard Brownian motion. The linearized SDE (L-SDE) is derived from Eq.(1) by performing a first-order Taylor series expansion to its drift term as

$$\mathrm{d}V = \psi\left(\overline{V} - V\right)\mathrm{d}t + \sigma \mathrm{d}B \tag{2}$$

where $\psi = 2C\overline{V}$ is the dumping coefficient and $\overline{V}$ is the cross sectionally-averaged flow velocity given by

$$\overline{V} = \sqrt{F / C}\ . \tag{3}$$

3. PDFs AND STATISTICAL MOMENTS

The Kolmogorov's forward equation (KFE) associated with the NL-SDE(1) is given by

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial v}\left[\left(F - Cv\left|v\right|\right)p\right] - \frac{\partial^2}{\partial v^2}\left(\frac{\sigma^2}{2}p\right) = 0 \tag{4}$$

where p is the conditional PDF, and v is the independent variable corresponding to the stochastic process V. The KFE(4) has a nontrivial stationary solution, which can be numerically computed. The KFE associated with the L-SDE(2) is

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial v}\left[\psi\left(\overline{V} - v\right)p\right] - \frac{\partial^2}{\partial v^2}\left(\frac{\sigma^2}{2}p\right) = 0\ , \tag{5}$$

whose nontrivial stationary solution is

$$p_{\mathrm{L}} = \sqrt{\frac{\psi}{\pi\sigma^2}}\exp\left[\frac{-\psi}{\sigma^2}\left(\overline{V} - v\right)^2\right]. \tag{6}$$

Numerical comparisons of the PDFs and the statistical moments of both of the models use the non-dimensional variables

$$W = V / \lambda \quad \text{and} \quad \overline{W} = \overline{V} / \lambda \tag{7}$$

where λ is the standard deviation. The PDFs and the statistical moments are calculated with the cell-vertex finite volume method[2)].

Fig.1 shows profiles of non-dimensionalized PDFs of the two models. The PDF of the L-SDE is the Gaussian distribution given by Eq.(6), which is plotted as a black line. The PDF does not explicitly depend on $\overline{W}$. The PDFs of NL-SDE are plotted for the different values of $\overline{W}$ as colored lines in the figure. **Fig.2** shows the statistical moments of the two models. The statistical moments of the L-SDE are plotted as dotted lines. The 1st

$\overline{W}$

and 3rd statistical moments of the L-SDE are 0, the 2nd moment is 1, and the 4th moment is 3. The statistical moments of the NL-SDE are plotted for the different values of $\overline{W}$ as colored lines. **Figs.1** and **2** show that the profiles of the PDFs and the moments of the NL-SDE approach those of the L-SDE as $\overline{W}$ increases.

According to the measured data of the Eulerian velocity time series of stationary turbulence in a real open channel[3)], the value of $\overline{W}$ is 7.4, and the 3rd moment of the data is negative, which is in accordance with the result of **Fig.2**. Under this flow condition, the differences between the non-dimensional values of the 1st and 2nd moments of the two models are less than 0.1.

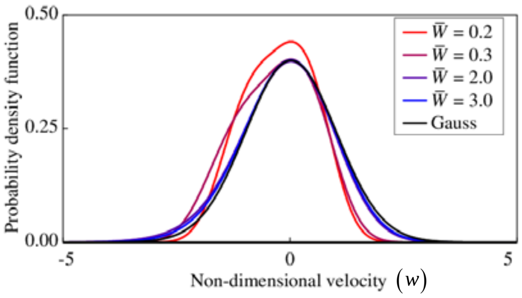

Fig.1 Non-dimensional PDFs of the two models

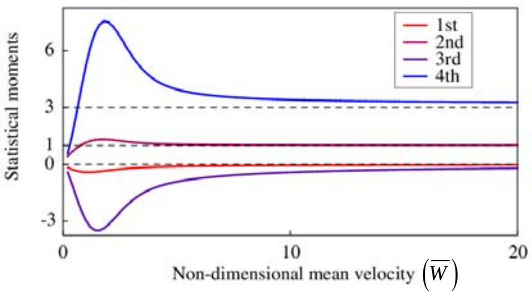

Fig.2 1st through 4th non-dimensional statistical moments of the two models

4. DISPERSION COEFFICIENTS

In order to analyze magnitude of dispersion, the kinetic equation of the Lagrangian particle movement is

$$\mathrm{d}X = V\mathrm{d}t \tag{8}$$

where X is the position of solute particle. The coupled model consisting of Eqs.(1) and (8) is referred to as the NL-SDEs. Similarly, the coupled model consisting of Eqs.(2) and (8) is referred to as the L-SDEs. In these models, the assumption that solute particles move along with the flow was set. The dispersion coefficient D effectively characterizing stochasticity of the particle movements in the flows is defined by

$$D = \lim_{t\to\infty} \frac{1}{2t} \mathrm{E}\left[(X - \overline{X})^2 \right] \tag{9}$$

where $\mathrm{E}[\cdot]$ is the expectation operator. The non-dimensional dispersion coefficient of the L-SDE using Eq.(7) is equal to 2, while that of the NL-SDE cannot be analytically obtained and is therefore computed with a particle method based on the Box-Muller's method for generating Gaussian random variables and the 4th order Runge-Kutta method for performing temporal integration.

Fig.3 presents the non-dimensional dispersion coefficient as a function of $\overline{W}$ based on 10,000 particle path realizations. **Fig.3** shows that the value of the non-dimensional dispersion coefficient of the NL-SDEs approaches 2 as $\overline{W}$ increases. By means of the same data[3)], the differences between the non-dimensional dispersion coefficients of the two models is calculated, which is less than 0.1.

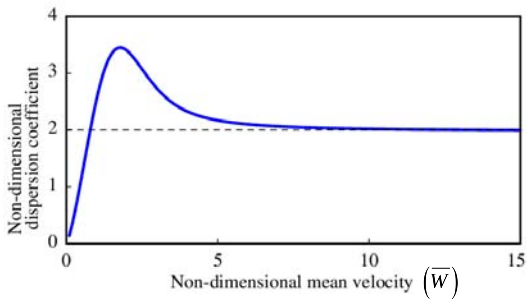

Fig.3 Non-dimensional dispersion coefficients

5. CONCLUSIONS

Mathematical and numerical analyses on the PDFs, the statistical moments, and the dispersion coefficients of the NL- and L -SDEs for Lagrangian movements of individual solute particles were performed. The results of the analyses using the non-dimensional variables indicated the ranges of the applicability of the linearization procedure to derive the L-SDE from the NL-SDE. The linearization procedure can be justified for the flows where $\overline{W}$ is sufficiently large. It should be noted that the present research results do not directly validate the models, which is one of the important research topics to be investigated in future researches.

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