京都大学全学共通講義　　平成２１年度前期

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第３回　分岐現象と安定性

Local bifurcations

1. Saddle-node bifurcations

In the present study, it is relying on a geometrical grasp of the phase portrait, rather than on complex analytic computations.

The vector field near a saddle-node bifurcation can be expressed as

$$x' = \mu - x^2$$

where x is a variable and μ is a control parameter.. The equilibrium equation (x' = 0) gives a quadratic relationship between μ and x, and the diagram in the plane (x, μ) shows a parabolic equilibrium path as shown in the following figure. One of the branches of this parabola is unstable, and marks the separation between the basin of attraction of the stable path and the basin of attraction of the point at infinity..

$x' = \mu - x^2$

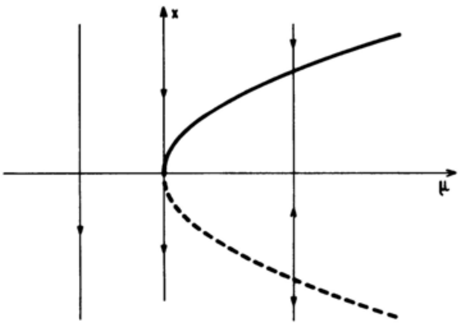


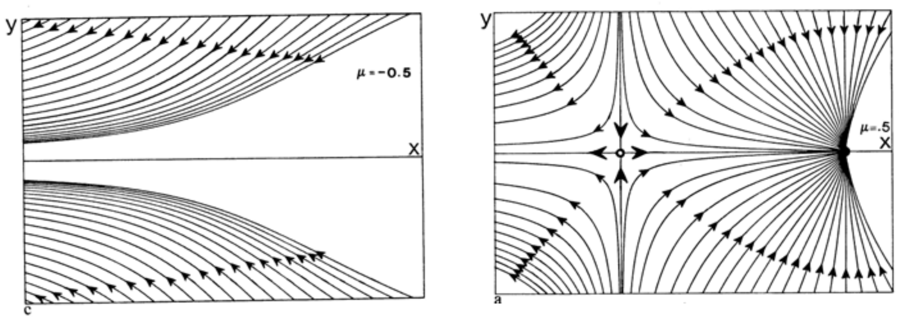
Figure A.2 Saddle-node bifurcation diagram.
The fixed point is located at the origin ($x = 0$, $\mu = 0$). Two branches of steady states emerge from the bifurcation point: one stable (heavy solid line), the other unstable (dashed line). The vertical lines represent the lines of force of the vector field, the arrows indicating the direction.

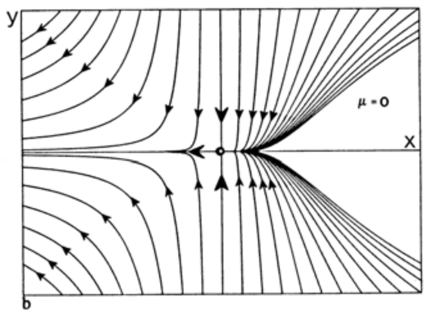
Pierre Berge, Yves Pomeau and Christian Vidal
“Order within Chaos” Wiley 1984, p274

The steady solution is defined only for $\mu > 0$, first appearing at $\mu = 0$. There exists no solution, stable or unstable, for $\mu < 0$.
It is useful to consider the above diagram embedded in a two-dimensional system. As a possible lifting of the one-dimensional equation to a two-dimensional space, let us consider two decoupled equations:

$$x' = \mu - x^2$$

$$y' = -y$$





$$x' = \mu - x^2$$
$$y = -\ y$$

Phase portraits for the saddle-node Bifurcation

J. M. T. Thompson and H. B. Stewart
"Nonlinear Dynamics and Chaos"
Wiley, 1986, p118

2. Transcritical bifurcatiions

Instead of the general saddle-node bifurcation, many physical systems exhibit transcritical bifurcations. The simple form is

$$x' = \mu\ x - x^2$$

As shown in the next figure, two steady solutions $x = 0$ and $x = \mu$ coexist. $x = 0$ is stable if $\mu < 0$ and unstable if $\mu > 0$, and vice versa for the $x = \mu$ solution. There is exchange of stability between the two solutions at the critical point. Owing to the form of the diagram this bifurcation is called an asymmetric point of bifurcation.

$x' = \mu x - x^2$

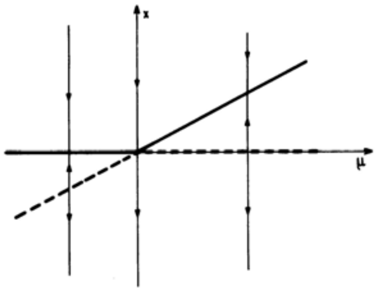
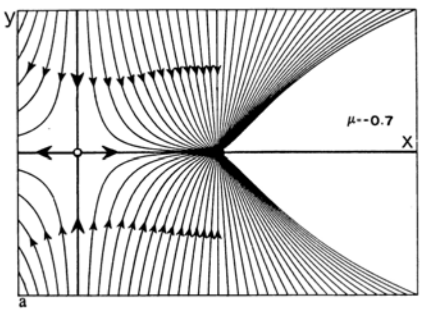


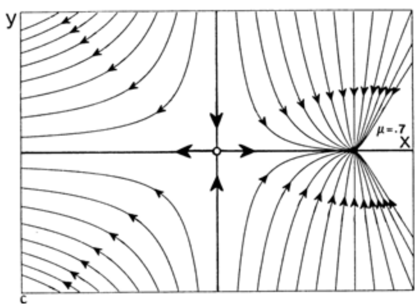
Figure A.3 Diagram of a transcritical bifurcation.

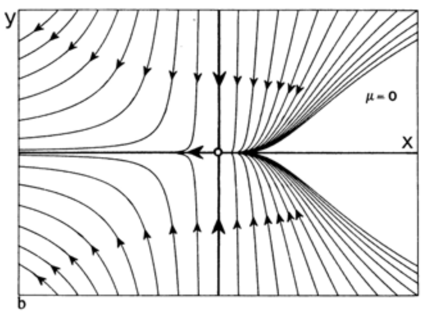
Pierre Berge, Yves Pomeau and Christian Vidal
"Order within Chaos" Wiley 1984, p274

It is again possible to lift the flow, in much the same way as for the saddle-node bifurcation, by means of the equations

$$x' = \mu x - x^2$$
$$y' = -y$$







$x' = \mu x - x^2$
$y = - y$

Phase portraits for the saddle-node Bifurcation

J. M. T. Thompson and H. B. Stewart
"Nonlinear Dynamics and Chaos"
Wiley, 1986, p123

The phase portrait of the middle panel is the same as for the saddle-node, because the governing equation is the same. This suggests that the only difference between the two bifurcations is the route through which the critical point is achieved.

3. Pitchfork bifurcation and distorted pitchfork bifurcation

If the system has some intrinsic symmetry, the transcritical bifurcation is no longer possible, and different kind of bifurcation appears. Indeed, the following system shows that the sink at the origin becomes a saddle and gives rise (in its supercritical form) to two symmetric sinks. Such a system can be described by

$$x' = \mu x - x^3$$

The bifurcation diagram is shown as follows.

$x' = \mu x - x^3$

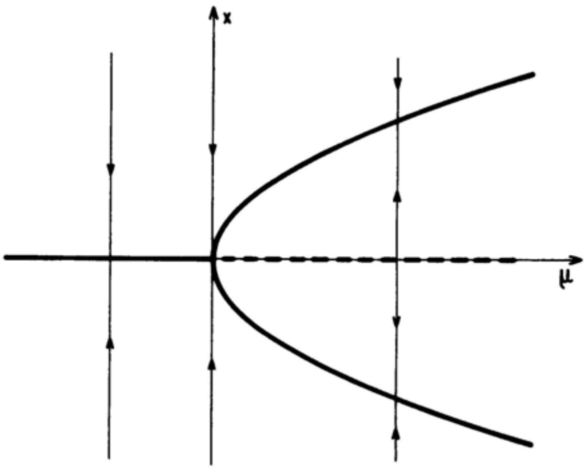


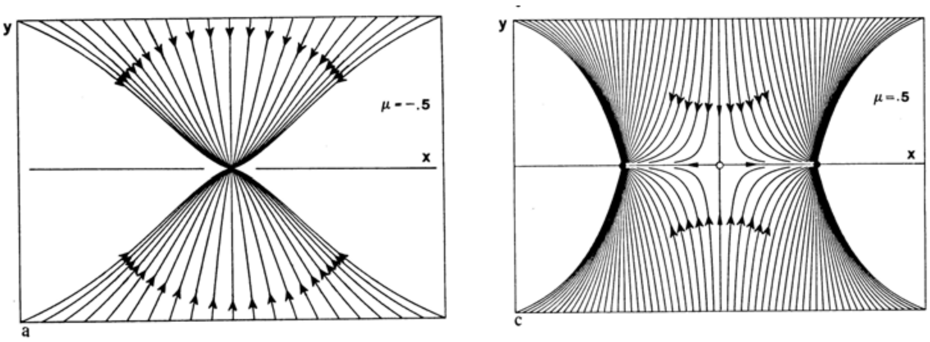
Figure A.4
Diagram of a pitchfork bifurcation.

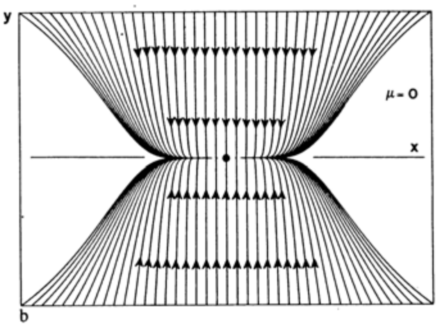
Pierre Berge, Yves Pomeau and Christian Vidal
“Order within Chaos” Wiley 1984, p275

The form of the diagram justifies the classical name of pitchfork bifurcation or symmetric point of bifurcation.

It is also possible to have the two-dimensional system

$$x' = \mu x - x^3$$
$$y' = -y$$



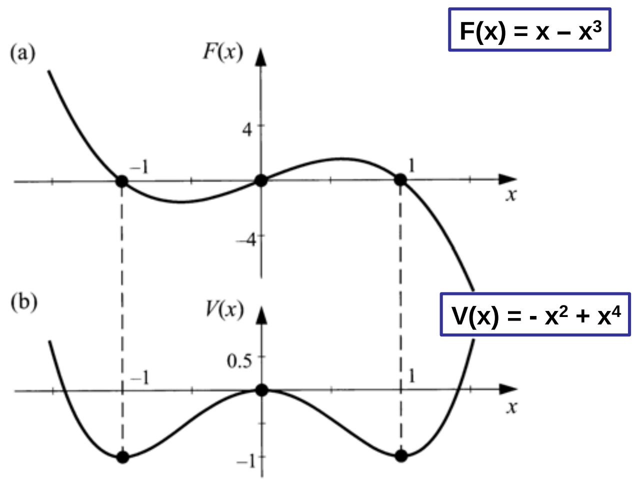


$$x' = \mu x - x^3$$
$$y = - y$$

Phase portraits for the saddle-node Bifurcation

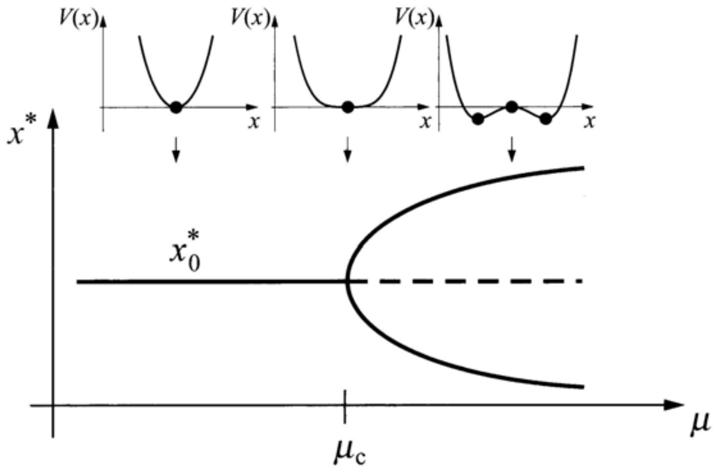
J. M. T. Thompson and H. B. Stewart
"Nonlinear Dynamics and Chaos"
Wiley, 1986, p124

In the first panel, there is only a sink at the origin, and all the trajectories tend towards it. When the parameter is set to be the critical value, every trajectory is attracted strongly to the line x axis, and the subsequent dynamics is essentially restricted to this one-dimensional subspace. The resulting subspace is usually called the centre manifold. When the parameter exceeds the critical value, the phase portrait is divided into two basins of attraction by the stable manifold of the saddle.



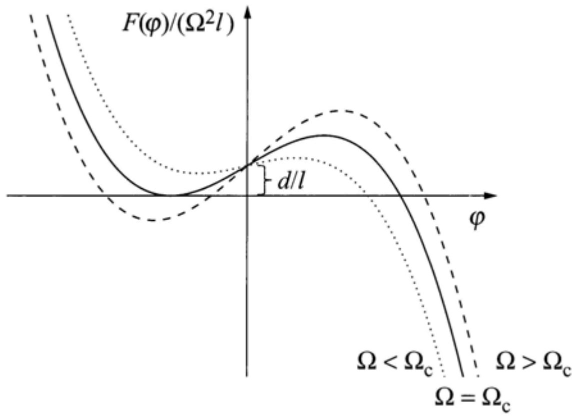
Tamas Tel and Marton Gruiz "Chaotic Dynamics"
Cambridge University Press 2006, p69

Fig. 3.19. Bifurcation diagram (schematic) of a pitchfork bifurcation. Co-ordinates x^* of the equilibrium states are given in terms of a parameter μ. The dashed line represents the unstable state emerging from a stable state. The shape of the potential is also plotted for selected values of the parameter.

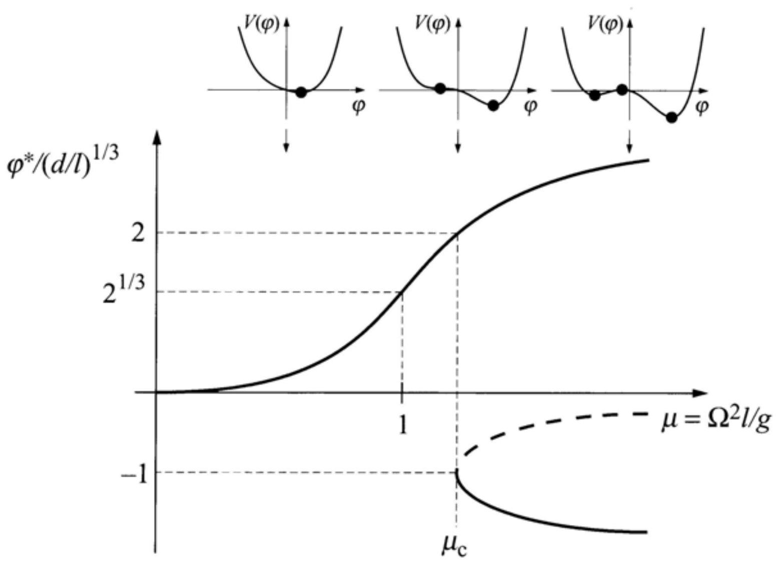


Tamas Tel and Marton Gruiz "Chaotic Dynamics" Cambridge University Press 2006, p74

In systems without symmetry, bifurcations occur in a different way. In such cases the stable state does not cease as the parameter changes, but rather two new equilibrium solutions appear above a critical value: one stable and one unstable. When decreasing the parameter from large values, one of the stable states suddenly disappears. This phenomenon is often called a catastrophe. The whole bifurcation process is a distorted pitchfork bifurcation.



Tamas Tel and Marton Gruiz "Chaotic Dynamics" Cambridge University Press 2006, p348

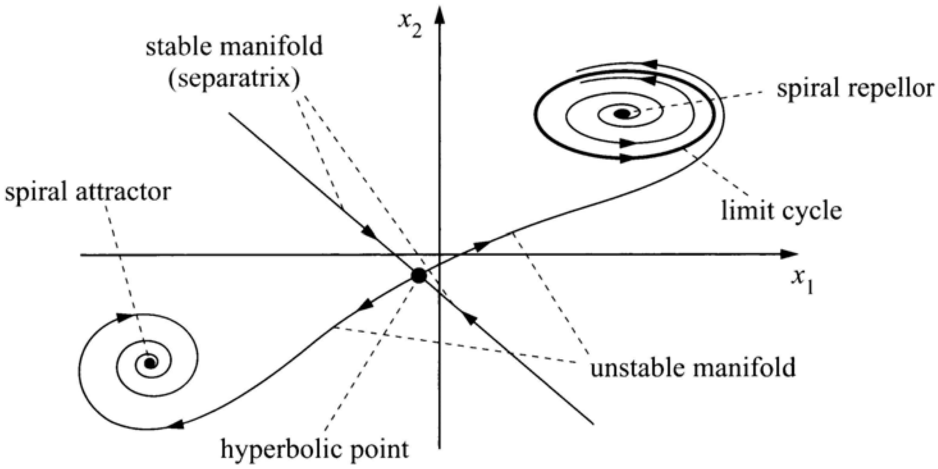


In asymmetrical systems of a distorted pitchfork bifurcation, at a critical value μ_c of the parameter, a second stable state arises abruptly, together with an unstable one (dashed line).

Tamas Tel and Marton Gruiz "Chaotic Dynamics" Cambridge University Press 2006, p348

Phase portraits of general two-dimensional flows

Determining fixed points and their stability, a complete phase portrait can be constructed. The stable and unstable manifolds of the saddle points play an important role. Both manifolds can be infinitely long curves extending over large domains of phase space. The stable manifold always separates different attractors, and it is the basin boundary; the unstable manifold traces out the way towards the attractors. In two-dimensional phase spaces there is no need for an extra search for limit cycle attractors, since they can be obtained by simply tracking the unstable manifolds. A qualitative knowledge of these manifolds yields an overall geometrical view of the dynamics, or even of its parameter-dependence, without having to solve the problem in detail. This approach will also be useful in studying chaotic motion. (From Tamas Tel and Marton Gruiz "Chaotic Dynamics" Cambridge University Press 2006, p88)



Tamas Tel and Marton Gruiz "Chaotic Dynamics"
Cambridge University Press 2006, p88