Quantum Mechanics

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Chapter 3

The Time-Independent Schrödinger Equation

In the Schrödinger Equation

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t) = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

the potential often does not depend on time; i.e. $V(x,t) = V(x)$. An example of this would be the infinite square well potential.

$$V(x,t) = \begin{cases} 0 & -\frac{a}{2} < x < \frac{a}{2} \\ \infty & |x| \geq \frac{a}{2} \end{cases} \quad (a > 0)$$

When $V(x,t) = V(x)$, we can express the multivariable function $\Psi(x,t)$ as a product of a function of x and a function of t as follows.

$$\Psi(x,t) = \psi(x)\phi(t)$$

This technique is called "separation of variables". Let us substitute the above Ψ into the Schrödinger Equation.

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\psi(x)\phi(t) = i\hbar\frac{\partial\psi(x)\phi(t)}{\partial t}$$

$$\implies -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x)\phi(t) + V(x)\psi(x)\phi(t) = i\hbar\frac{\partial\psi(x)\phi(t)}{\partial t}$$

$$\implies -\frac{\hbar^2}{2m}\phi(t)\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x)\phi(t) = \psi(x)i\hbar\frac{\partial\phi(t)}{\partial t}$$

$$\Longrightarrow \phi(t)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) + V(x)\psi(x)\right] = \psi(x)i\hbar\frac{\partial\phi(t)}{\partial t}$$

Dividing through by $\psi(x)\phi(t)$, we get

$$\underbrace{\frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x)\right]}_{H(x)} = \underbrace{i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt}}_{K(t)};$$

where we changed the partial derivatives to total derivatives as we now have one-variable functions.

Generally speaking,

$$H(x) = K(t) \text{ for all pairs } (x, t)$$

means that both $H(x)$ and $K(t)$ are a constant. Let $H(x) = K(t) = G$ (a constant). Then,

$$K(t) = G \Longleftrightarrow i\hbar\frac{1}{\phi(t)}\frac{d\phi(t)}{dt} = G \Longleftrightarrow \frac{d\phi(t)}{dt} = \frac{1}{i\hbar}G\phi(t) = \frac{-i}{\hbar}G\phi(t).$$

Therefore,

$$\phi(t) = Ae^{-iGt/\hbar}.$$

Compare this with the time-dependent part of the travelling wave

$$e^{i(kx-\omega t)} = e^{ikx}e^{-i\omega t}.$$

We realize that $\frac{G}{\hbar} = 2\pi\frac{G}{h}$ "should be" ω.

$$2\pi\frac{G}{h} = \omega \Longleftarrow G = \frac{\omega h}{2\pi} = \frac{2\pi\nu h}{2\pi} = h\nu = E.$$

We now have $G = E$, and

$$\phi(t) = e^{-iEt/\hbar}.$$

This also implies

$$H(x) = \frac{1}{\psi(x)}\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x)\right] = E \Longrightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x)+V(x)\psi(x) = E\psi(x);$$

where E is the total energy.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = E\psi(x) \text{ or } \left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\psi(x) = E\psi(x)$$

is the Time-Independent Schrödinger Equation.

Note here that the full wavefunction is given by

$$\Psi(x,t) = \psi(x)e^{-iEt/\hbar}.$$

In the rest of this course, we will solve the Time-Independent Schrödinger Equation for various systems, culminating in a solution for the hydrogen atom.