International Workshop on
What is Evolution?
- Bicentennial of Charles Darwin's Birth -

A method for constructing databases for global dynamics of multi-parameter systems

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Growing interest in dynamics of

systems with large degrees of freedom

e.g. coupled systems, network dynamics, ...

Difficulties for understanding such systems

- Lack of useful theory

  Naive analysis is very limited

  General theory is often not very helpful

- Numerical simulation can give little information

  Phase space is too large

  Easily miss important part of dynamics

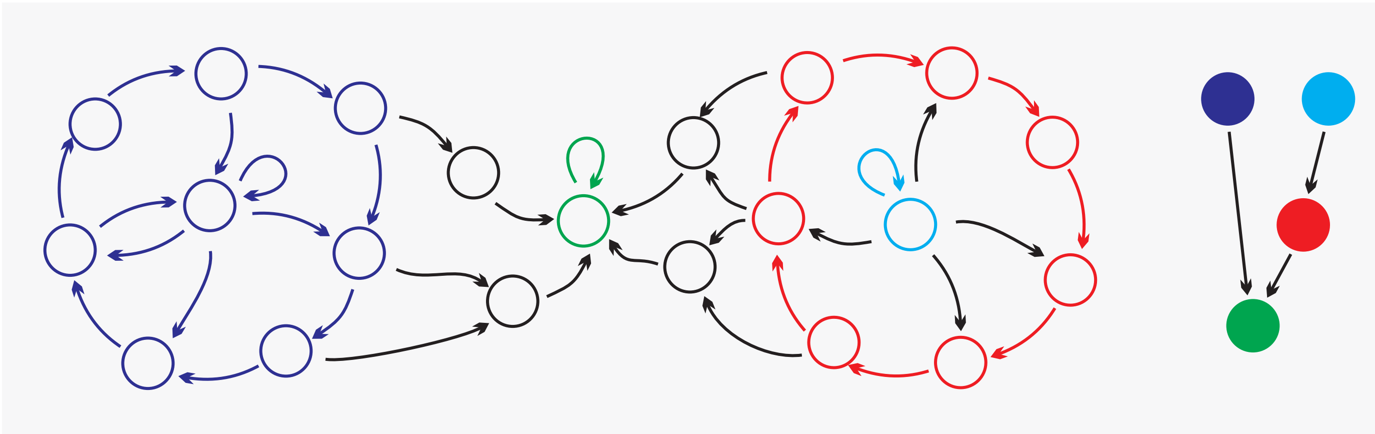
  Hard to capture global structure

  Too many parameters to control

Description of global dynamics, insensitive to dimension

Our approach:

Graph-based description of dynamical information



Features

- Rigorous "outer-approximation" of global dynamics
- Combination of Dynamics, Topology, and Computation
- Can construct a "Database" for
  dynamics of multi-dim, multi-parameter systems

Outline of the proposed method

(1) Grid decomposition of phase & parameter space

(2) RIgorous outer-approximation of dynamics

Interval arithmetic

(3) Graph representation of dynamics

(4) Gradient-like vs Recurrent decomposition of dynamics

Morse decomposition

(5) Topological representation for recurrent dynamics

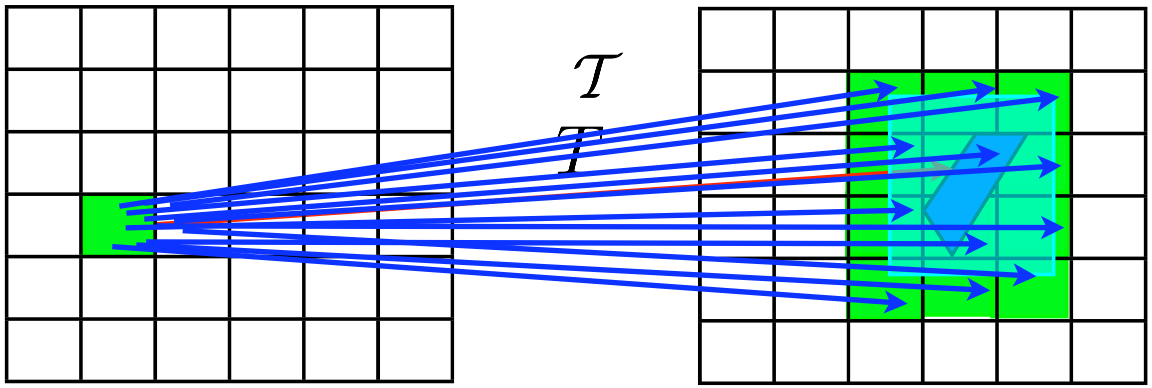
Conley index

(6) Collect all information and build a "database"

Rigorous combinatorial description of dynamics

Suppose a dynamical system given by iterates of a map T

Cubical grid decomposition of phase space



Rigorous error bound using interval arithmetic

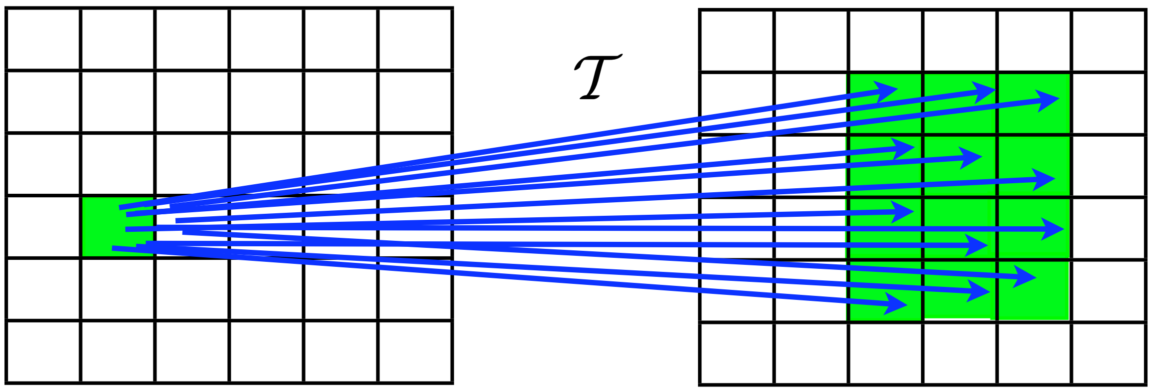
Combinatorial multi-valued map on cubical grid

Rigorous outer-approximation of the dynamics:

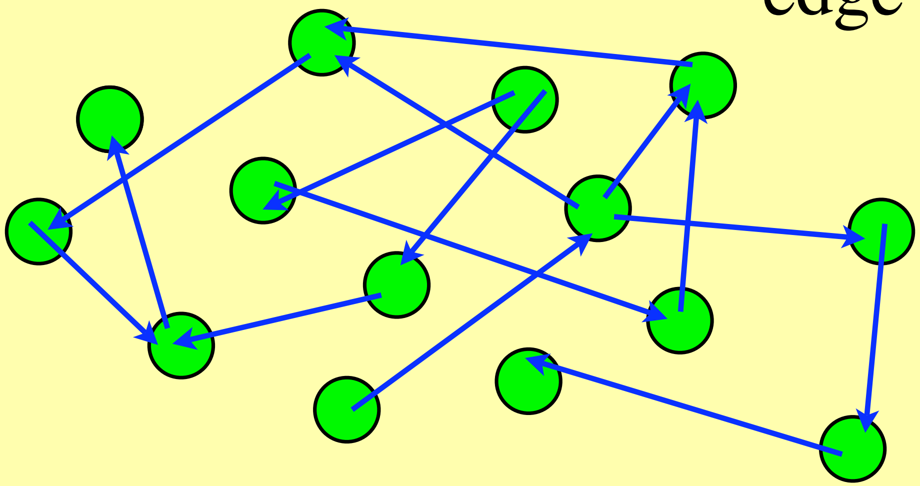
$$T(B) \subset \mathrm{int}|\mathcal{T}(B)| \ (\forall B)$$

Graph representation of dynamics

combinatorial multi-valued map



→ directed graph $\mathcal{G}$ with vertex = cube, edge = arrow as above



Task: obtain dynamical properties from the graph

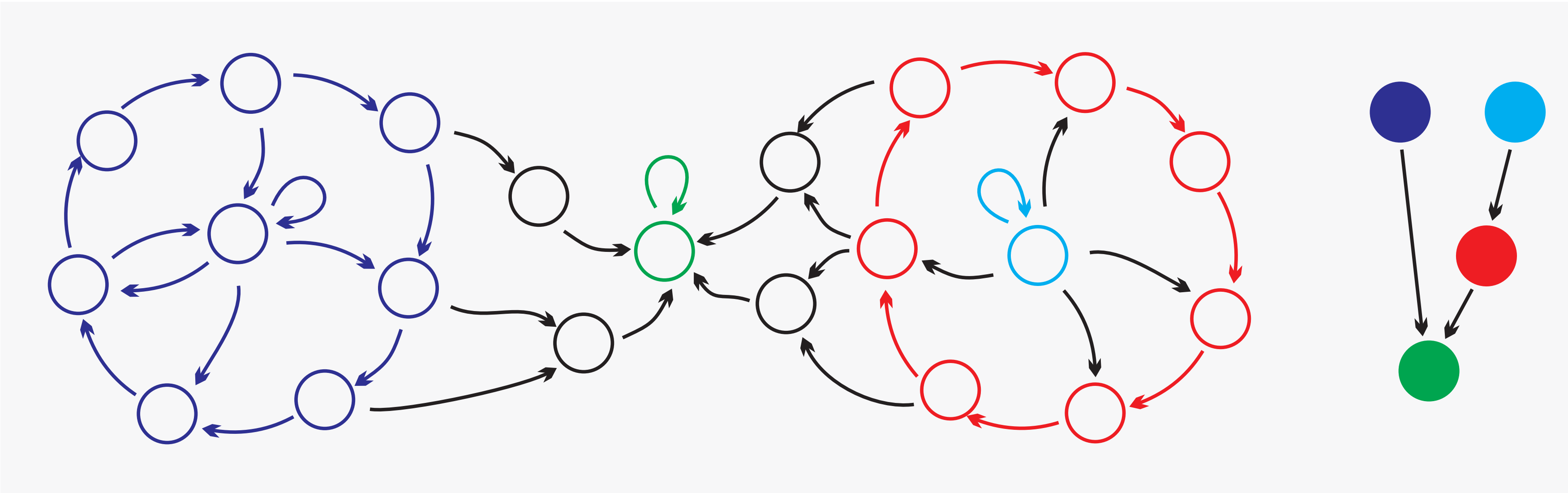
Combinatorial invariant sets [Kalies et al 2005]

Collection of all cubes with a bi-infinite path

$\mathrm{Inv}_f(N) \subset \mathrm{Inv}(\mathcal{G})$: combinatorial maximal invariant set

Collection of all cubes with a loop

$\mathcal{R}(f) \subset \mathrm{Scc}(\mathcal{G})$: combinatorial chain-recurrent set

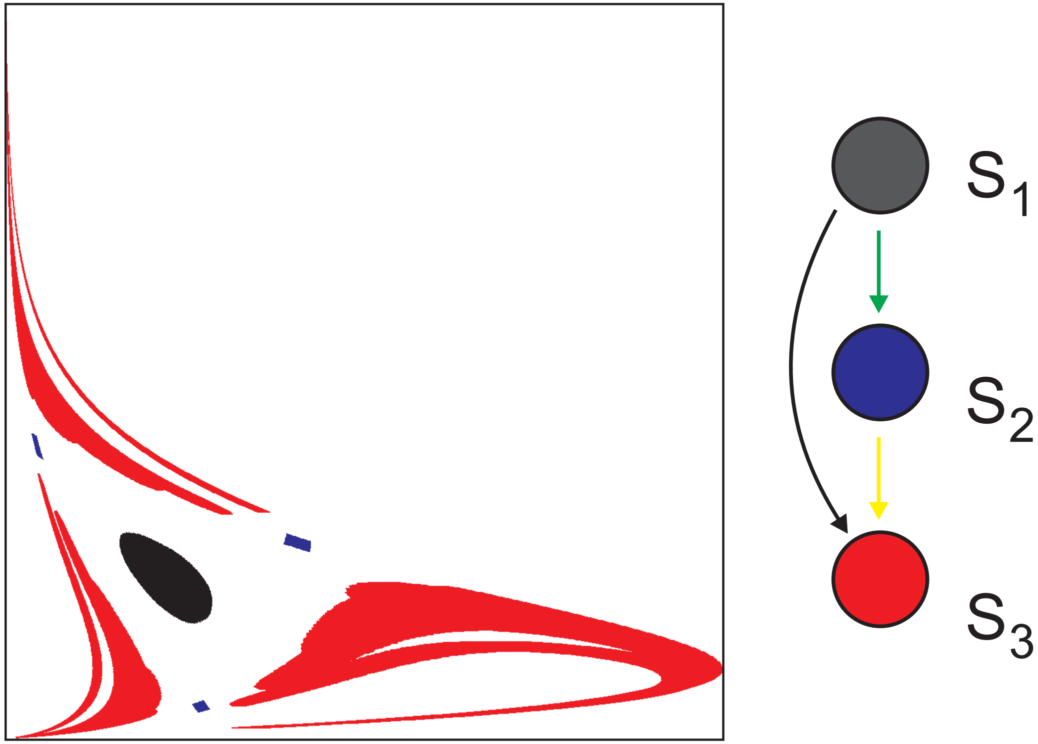


$\mathrm{Inv}(\mathcal{G}),\ \mathrm{Scc}(\mathcal{G})$ computable by fast graph algorithms

Combinatorial Morse decomposition [Ban-Kalies, 2006]

Dynamics gradient-like (or uni-directional)

outside *combinatorial chain-recurrent components*



Different colors represent different Morse sets
Combinatorial connecting orbits of the graph

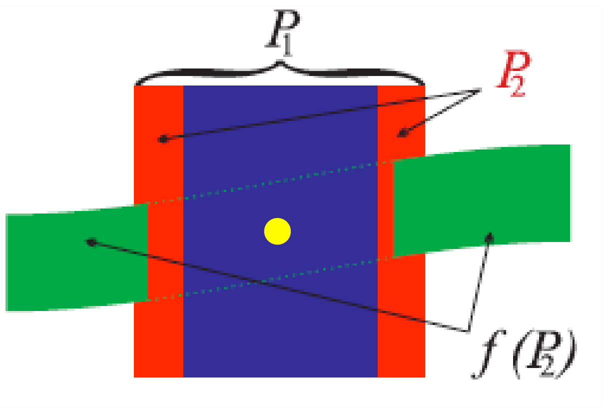
Warning: true connecting orbits might be empty

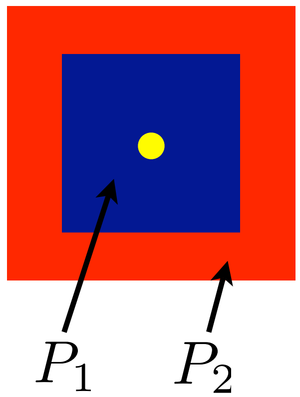
Partial topological information of recurrent dynamics

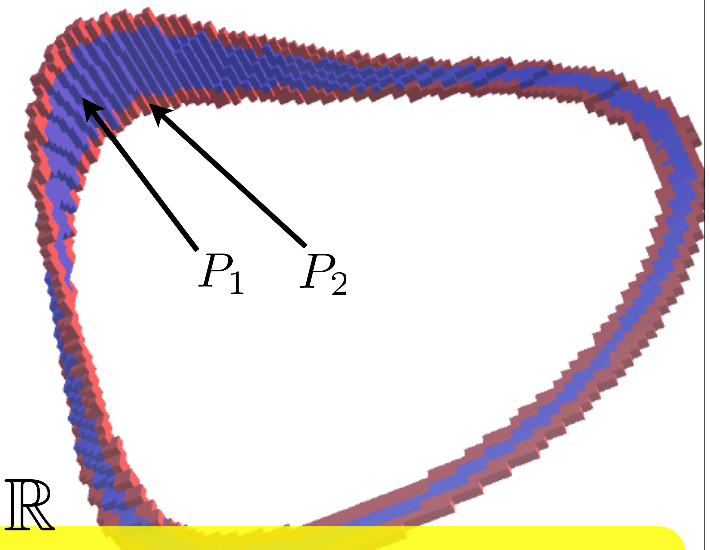
Conley index for an *isolated invariant set*
is the "*shift-equivalence class*"
of the *homology map* of an *index pair*

E.g.

Index pair (P_1, P_2) (P_1 : isolating nbhd ; P_2 : exit set)







$H_1(P_1, P_2) = \mathbb{R}$

$f_1(x) = \pm x$

$H_2(P_1, P_2) = \mathbb{R}$

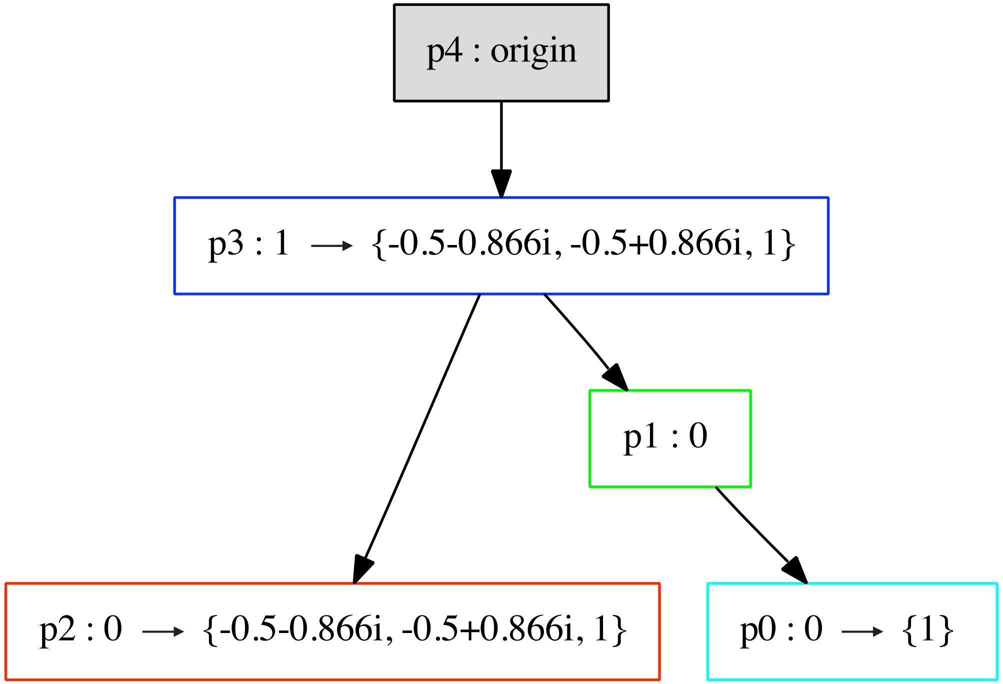
$f_2(x) = \pm x$

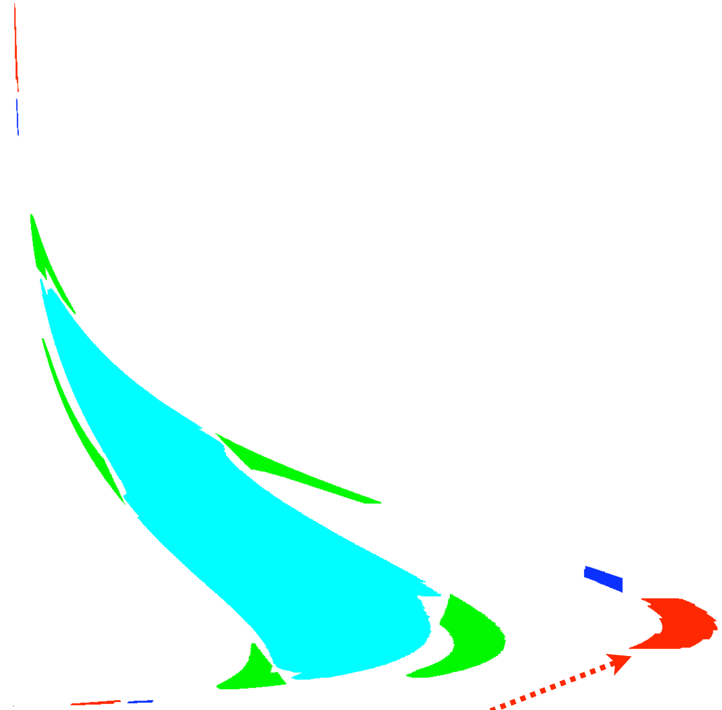
Can be obtained from combinatorial MV map

In practice: Conley index = degree + non-zero e.v.

Conley-Morse graph

Morse decomp. of phase space & Conley index





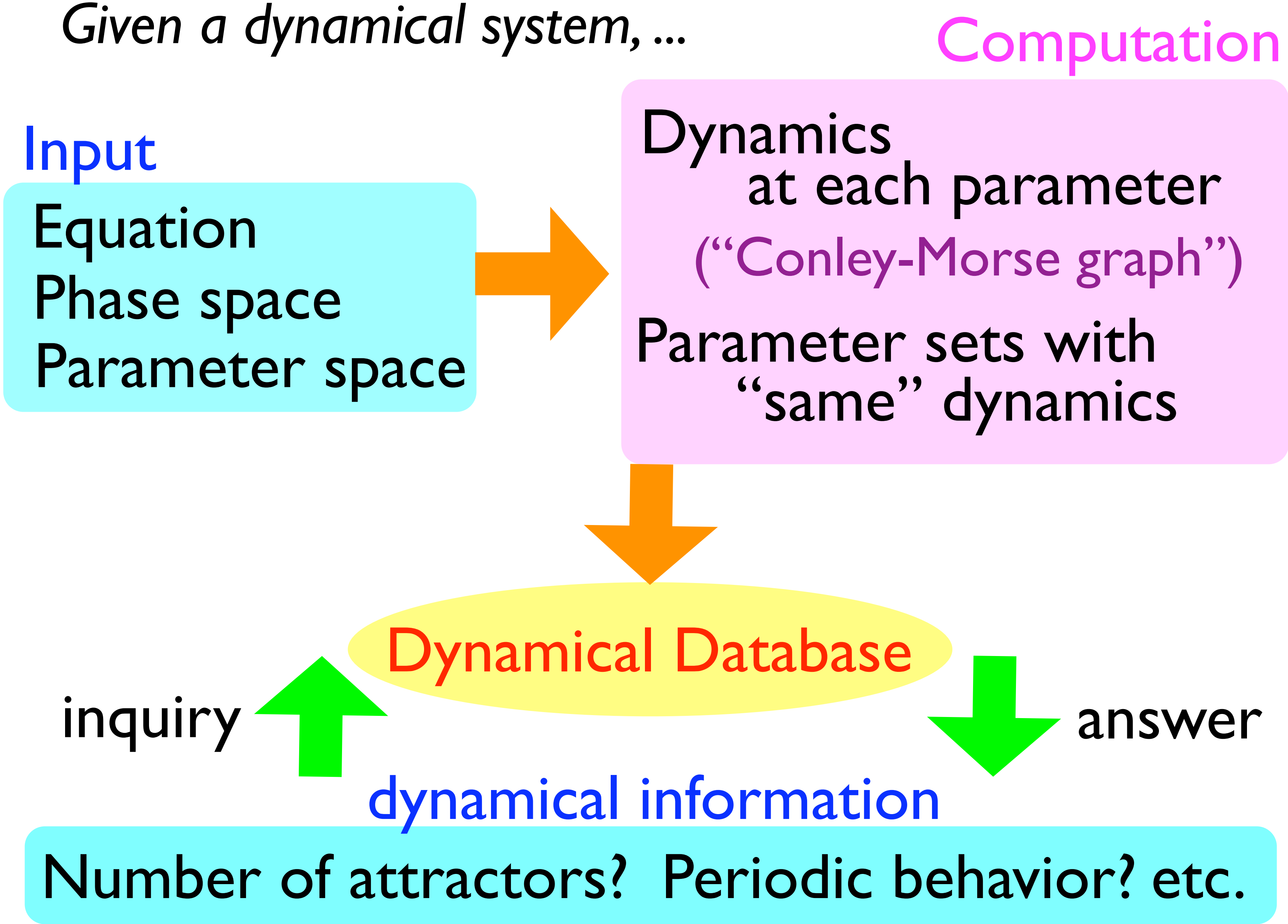
$$H_0(P_1, P_2) = \mathbb{R}^3$$

$$f_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

⇒ attracting 3-cycle set

Dynamical Database

Given a dynamical system, ...



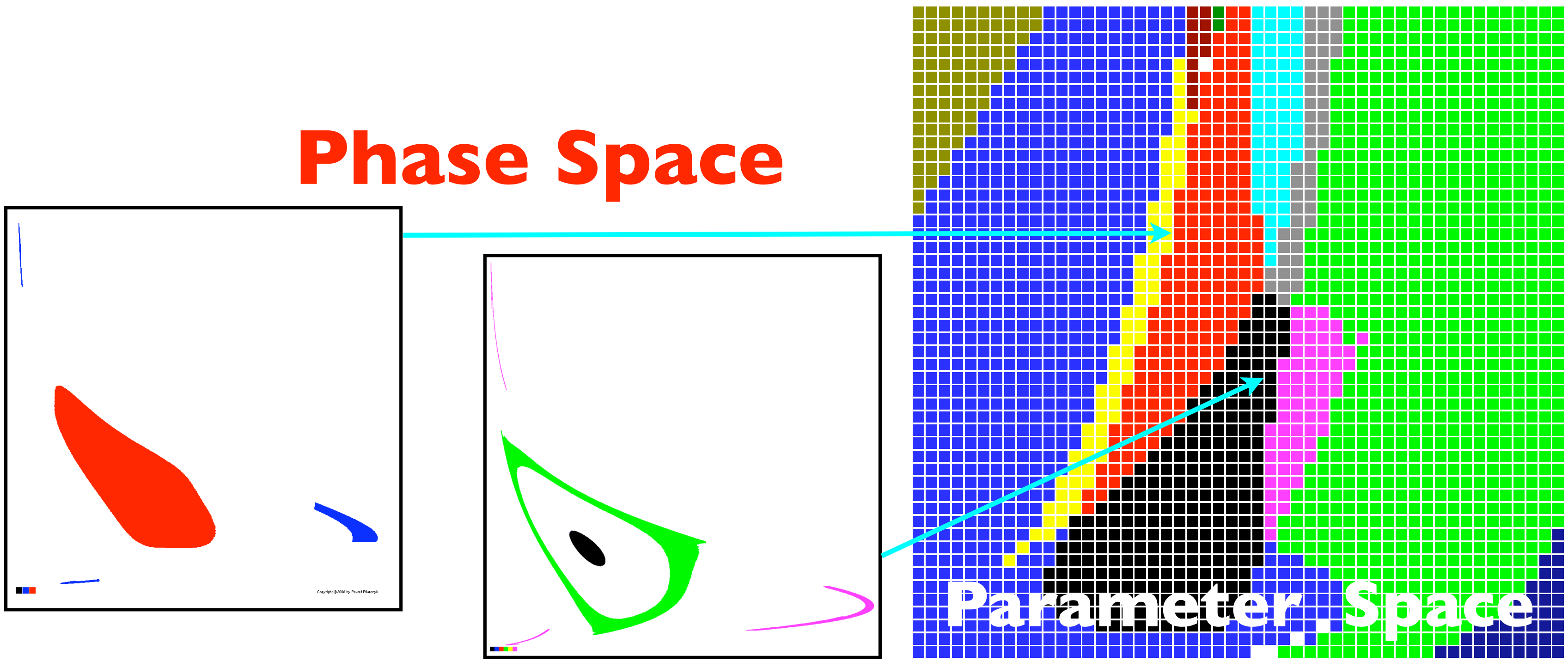
Dynamical Database: illustration

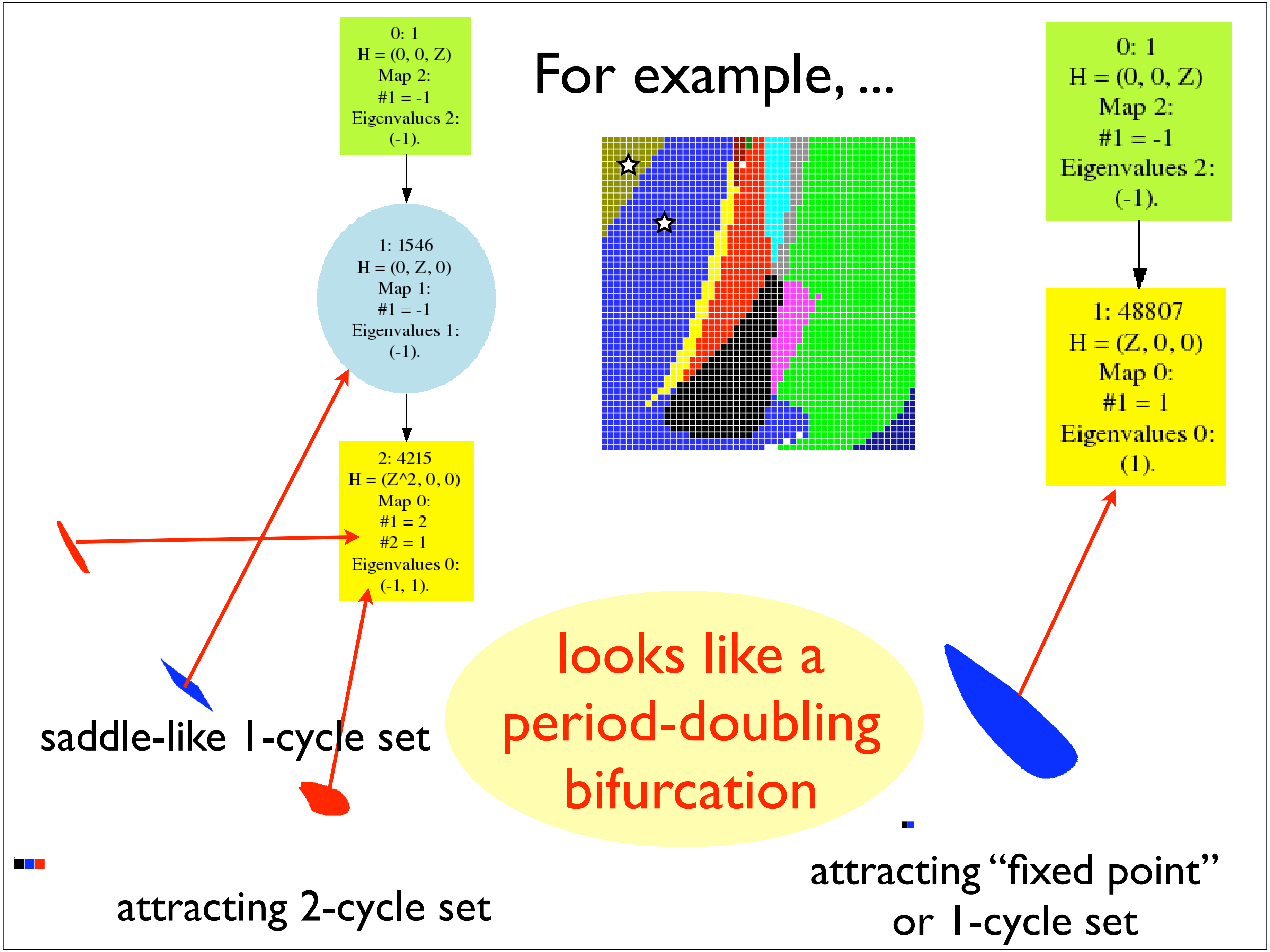
[Arai et al, SIADS 2009]

Nonlinear Leslie model [Ugarcovici-Weiss 2004]

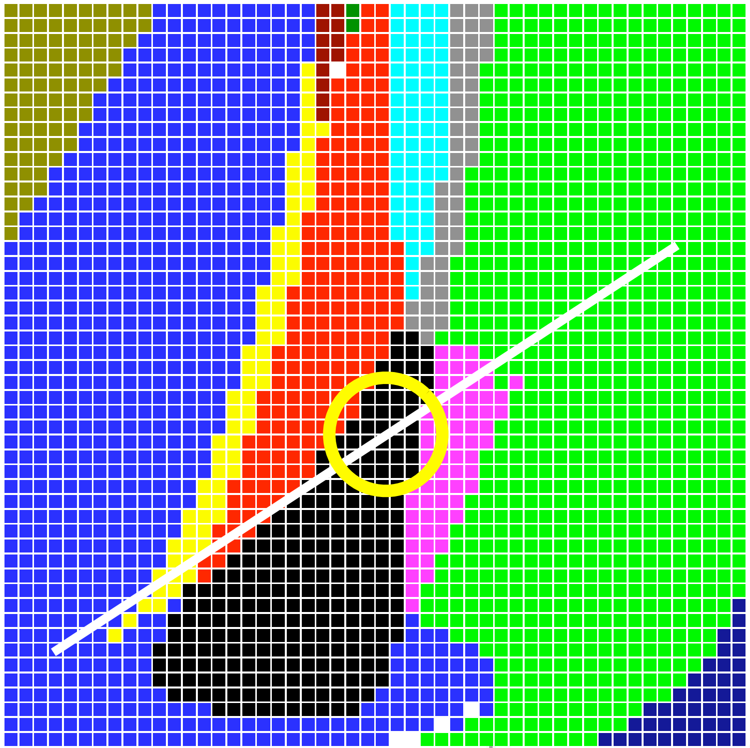
$$T : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (f_1 x_1 + f_2 x_2) e^{-0.1(x_1+x_2)} \\ p x_1 \end{pmatrix}$$

$$10 \leq f_1, f_2 \leq 50, \;\; p = 0.7$$

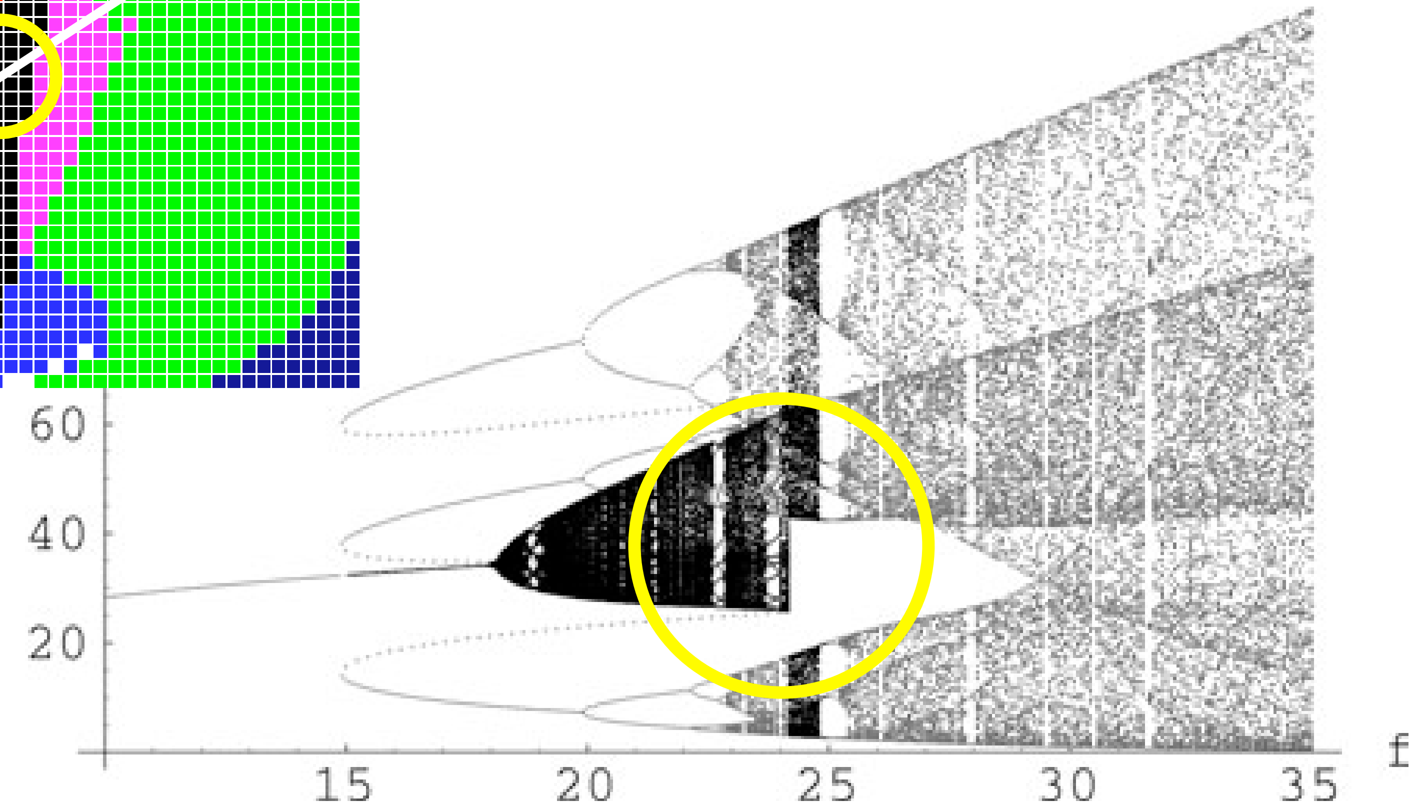



For example, ...
0: 1
H = (0, 0, Z)
Map 2:
#1 = -1
Eigenvalues 2:
(-1).
1: 1546
H = (0, Z, 0)
Map 1:
#1 = -1
Eigenvalues 1:
(-1).
2: 4215
H = (Z^2, 0, 0)
Map 0:
#1 = 2
#2 = 1
Eigenvalues 0:
(-1, 1).
0: 1
H = (0, 0, Z)
Map 2:
#1 = -1
Eigenvalues 2:
(-1).
1: 48807
H = (Z, 0, 0)
Map 0:
#1 = 1
Eigenvalues 0:
(1).
saddle-like 1-cycle set
looks like a period-doubling bifurcation
attracting 2-cycle set
attracting “fixed point” or 1-cycle set

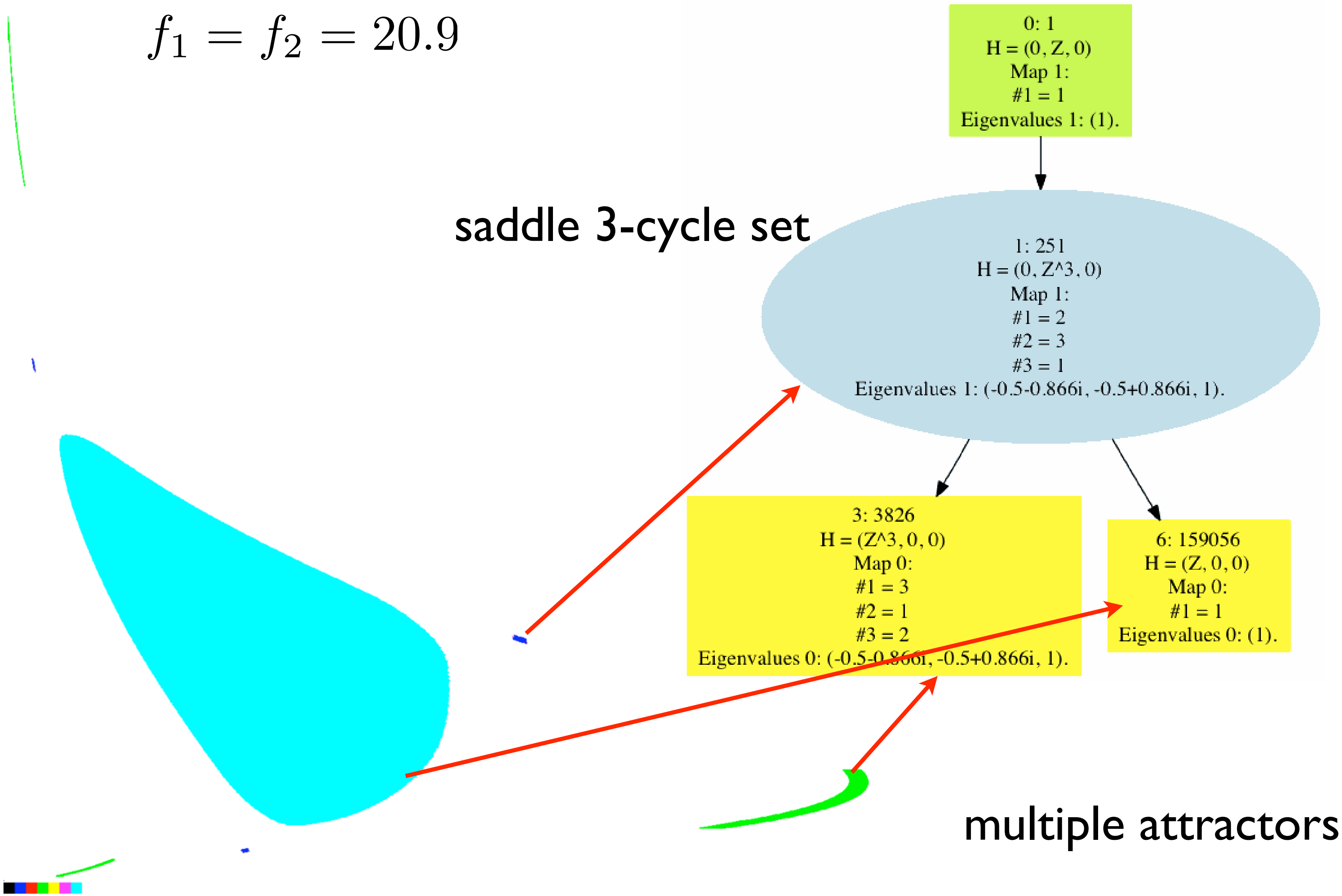
Comparison with usual numerical simulation



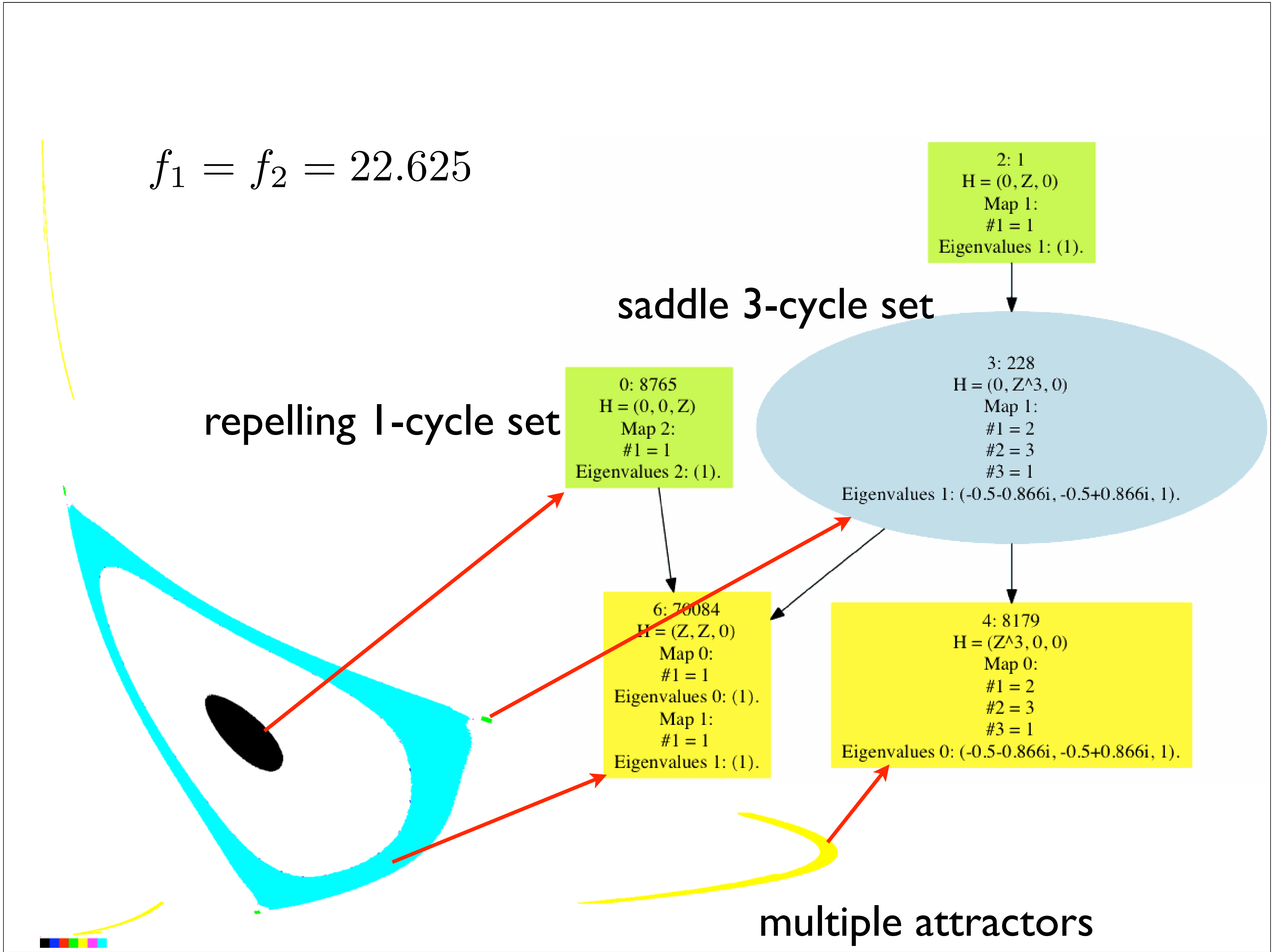
Bifurcation diagram in
[Ugarcovici-Weiss, 2004]



$$f_1 = f_2 = 20.9$$

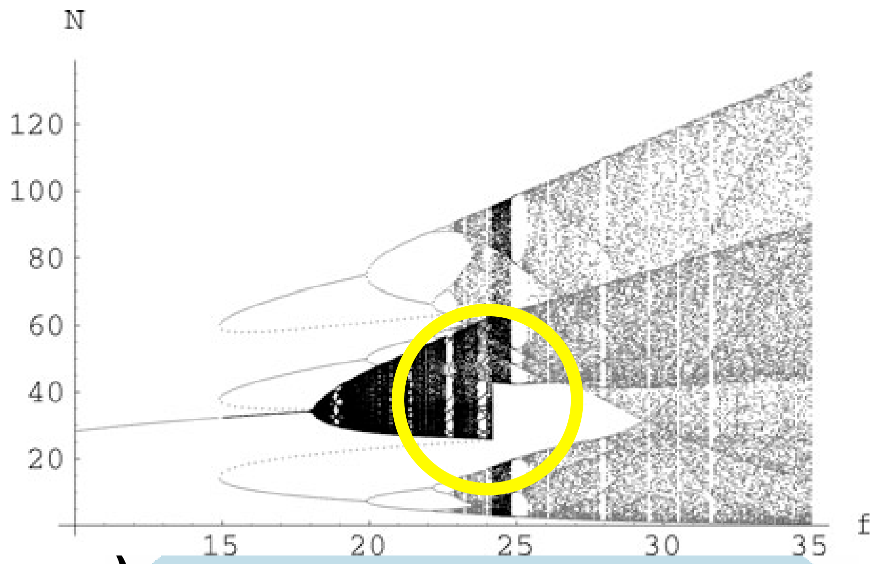


saddle 3-cycle set
0: 1
H = (0, Z, 0)
Map 1:
#1 = 1
Eigenvalues 1: (1).
1: 251
H = (0, Z^3, 0)
Map 1:
#1 = 2
#2 = 3
#3 = 1
Eigenvalues 1: (-0.5-0.866i, -0.5+0.866i, 1).
3: 3826
H = (Z^3, 0, 0)
Map 0:
#1 = 3
#2 = 1
#3 = 2
Eigenvalues 0: (-0.5-0.866i, -0.5+0.866i, 1).
6: 159056
H = (Z, 0, 0)
Map 0:
#1 = 1
Eigenvalues 0: (1).
multiple attractors



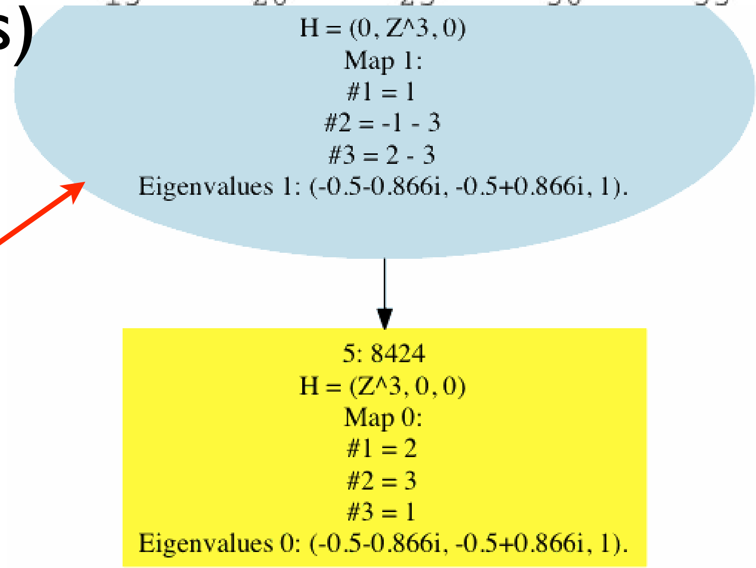
$f_1 = f_2 = 22.625$
saddle 3-cycle set
repelling 1-cycle set
multiple attractors
2: 1
H = (0, Z, 0)
Map 1:
#1 = 1
Eigenvalues 1: (1).
0: 8765
H = (0, 0, Z)
Map 2:
#1 = 1
Eigenvalues 2: (1).
3: 228
H = (0, Z^3, 0)
Map 1:
#1 = 2
#2 = 3
#3 = 1
Eigenvalues 1: (-0.5-0.866i, -0.5+0.866i, 1).
6: 70084
H = (Z, Z, 0)
Map 0:
#1 = 1
Eigenvalues 0: (1).
Map 1:
#1 = 1
Eigenvalues 1: (1).
4: 8179
H = (Z^3, 0, 0)
Map 0:
#1 = 2
#2 = 3
#3 = 1
Eigenvalues 0: (-0.5-0.866i, -0.5+0.866i, 1).

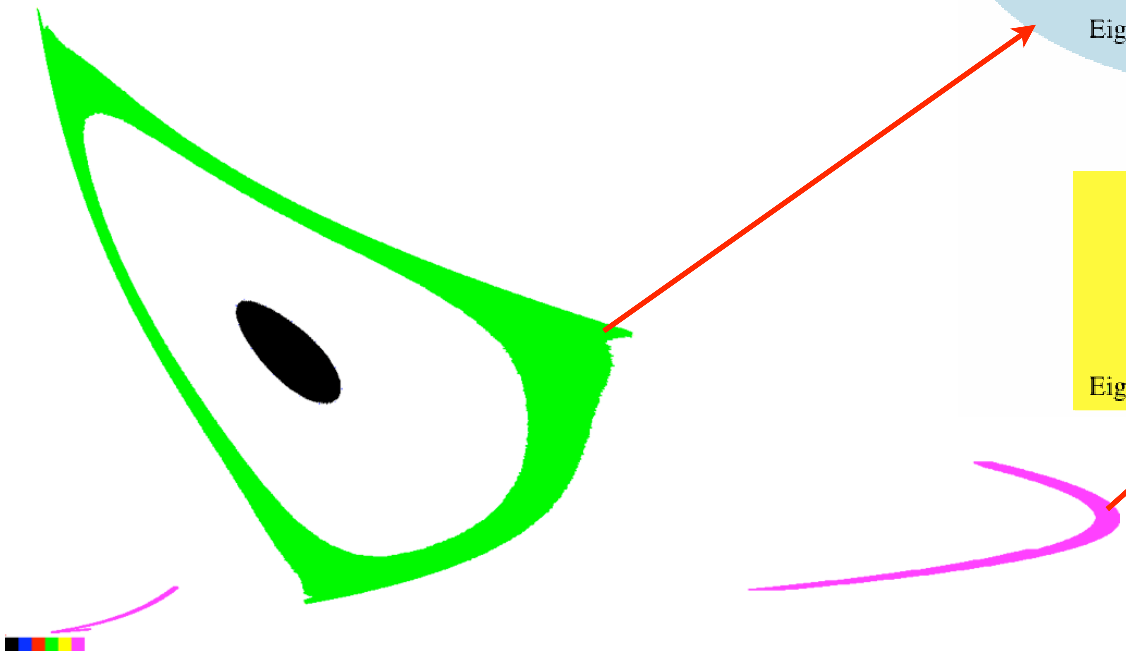
$$f_1 = f_2 = 22.825$$



Boundary Crisis

(one of attractors disappears)

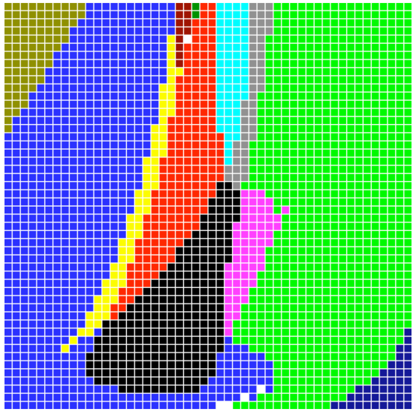


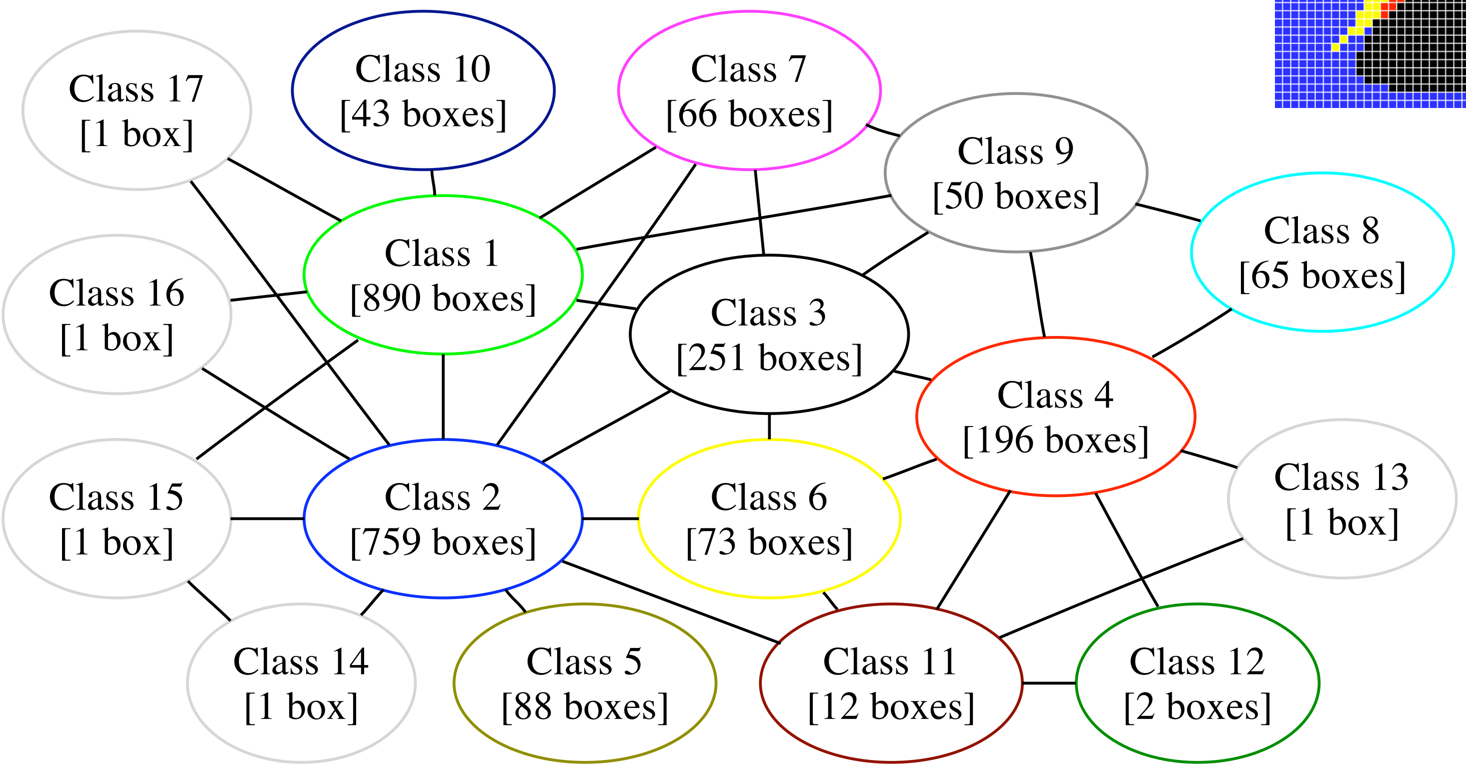


attracting 3-cycle set

Graph representation of Parameter sp. structure

Continuation Class & Continuation Graph





Some Future Problems

Computational problems

- Higher dim (phase and parameter) spaces
- Flow case (ODEs)
- Improvement of algorithms

Mathematical problems

- Better representation of dynamics
- Internal structure of recurrent sets
- How to identify bifurcations?

References

A.Zin, W.Kalies, H.K, K.Mischaikow, H.Oka, P.Pilarczyk,
SIAM Applied Dynamical Systems, 8 (2009), 757-789

(and references therein)

http://chomp.rutgers.edu/database

Interactive diagrams for computed results

3-parameter results

Links to source code and related software, etc.